THE FUNDAMENTAL LIMIT THEOREMS IN PROBABILITY

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1. **Introduction.** The main purpose of this address is to explain the mathematical content and meaning of the two most important limit theorems in the modern theory of probability: the central limit theorem[1] and the recently discovered precise form of what was generally known as "Kolmogoroff's celebrated law of the iterated logarithm." The former traces its origin to the very beginnings of the theory of probability and is often called after Laplace and Ljapunov. For a long time it was clouded in mystery, and Poincaré once remarked that mathematicians regard it as a physical law, whereas physicists hold mathematicians responsible for it. A great many mathematicians have contributed to the gradual recognition of the mathematical content of the theorem and to the establishment of the precise conditions of its validity. The complete solution came finally in 1935 and was possible only by an elimination of all classical restrictions and a reconsideration of the problem in a new generality.

The central limit theorem (like its little brother, the weak law of large numbers) is a statement on distribution functions, and can be formulated, either as such or in terms of Fourier analysis, without any appeal to probability or measure. This is not true of the infinitely more delicate law of the iterated logarithm and its generalizations (or of the strong law of large numbers): these are essentially measure-theoretic. The starting point of the long series of papers which lead to the present form of the iterated logarithm was not a problem in probability but, surprisingly enough, a problem in Diophantine approximations treated by Hardy and Littlewood [1914].[2] Their original estimate has gradually been improved for their particular number-theoretical case and, as a matter of fact, even the precise form of the iterated logarithm has first been checked for this particular case. It is therefore instructive to realize that, from the point of view of the general theory, the Hardy-Littlewood problem constitutes an exceedingly special case comparable only to the role of the linear function within the domain of all real functions. Such special

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[1] The name "central limit theorem" is due to Pólya [1920].

[2] Author's names, and years, appearing in brackets refer to the references cited at the end of the paper.