PERMUTATIONS WITHOUT 3-SEQUENCES

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1. **Introduction.** The enumeration of permutations of n distinct elements without rising 2-sequences 12, 23, $\cdots$, $n-1$ n, is given by Whitworth [1],[1] who gives also the enumeration when $n1$ is added to this set of sequences. More recently, Kaplansky [2] and Wolfowitz [4] have enumerated permutations without rising or falling 2-sequences, that is, without 21, 32, $\cdots$, n $n-1$ as well as 12, $\cdots$, $n-1$ n. An addition to these results, the enumeration of permutations without 3-sequences, 123, $\cdots$; $n-2$ $n-1$ n is given here. This case, aside from its general interest as a natural extension of its predecessors, has a particular interest because it is a relatively simple example of failure of what Kaplansky has called quasi-symmetry. In the method of inclusion and exclusion, or its symbolic equivalent, a case is said to be quasi-symmetric when the number of permutations having k of the given properties is either zero or a function of k alone.

2. **The enumeration setting.** Employing the symbolic method, with q_{ijk} the probability that elements i, j and k are consecutive, the probability of finding a permutation without any of the 3-sequences in question is

$$P_0 = (1 - q_{123})(1 - q_{234}) \cdots (1 - q_{n-2\ n-1\ n}). \tag{1}$$

The meaning of this is that on expansion a product of q's represents the probability of permutations having a particular set of 3-sequences denoted by subscripts of these q's.

It is evident that the sequences chosen do not conflict; that is, it is possible to have any k of the $n-2$ simultaneously.

For a single q the number of permutations is $(n-3+1)!$, for 3 elements are required for the corresponding 3-sequence which may be permuted as a single entity; the corresponding probability is $(n-2)!/n!$.

For a product of two q's however, the case is otherwise. If the two are immediately adjacent, like 123, 234, the number of permutations is $(n-4+1)!$ or $(n-3)!$; if not, like 123, 345 or 123, 456, the number is either $(n-5+1)!$ or $(n-6+2)!$, in either case $(n-4)!$. If sequences are denoted by their initial numbers, the number of permutations is

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.