NOTE ON APPELL POLYNOMIALS

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An interesting characterization of Appell polynomials by means of a Stieltjes integral has recently been given by Thorne.[1] We propose to give a second such representation, and to extend the result to the case of sets of polynomials of *type zero*, of which Appell sets form a subclass.

Appell sets may be defined by either of the following equivalent conditions: $\{P_n(x)\}$, $n=0, 1, \cdots$, is an Appell set (P_n being of degree exactly n) if either

(i) $P_n'(x)=P_{n-1}(x)$, $n=1, 2, \cdots$,

or

(ii) there exists a formal power series $A(t)=\sum_0^\infty a_n t^n$ ($a_0 \neq 0$) such that (again formally)

$$A(t)e^{tx} = \sum_0^\infty P_n(x)t^n.$$

The function $A(t)$ may be called the *determining function* for the set $\{P_n(x)\}$. The essence of Thorne's result is the following:

THEOREM OF THORNE. *A polynomial set $\{P_n(x)\}$ is an Appell set if and only if there exists a function $\alpha(x)$ of bounded variation on* $(0, \infty)$ *with the following properties:*

(i) *The moment integrals*

$$\mu_n = \int_0^\infty x^n d\alpha(x)$$

all exist.

(ii) $\mu_0 \neq 0$.

(iii) $\int_0^\infty P_n^{(r)}(x)d\alpha(x)=\delta_{nr}$, $\delta_{nr}=1$ *for* $n=r$, $\delta_{nr}=0$ *for* $n\neq r$.

And for the set $\{P_n(x)\}$ the determining function $A(t)$ is given by

$$A(t) = \left[\sum_0^\infty \mu_n \frac{t^n}{n!}\right]^{-1} = \left[\int_0^\infty e^{tx}d\alpha(x)\right]^{-1}.$$

The Stieltjes integral characterization that we now give will be seen to be essentially different from that in (iii) above.

THEOREM 1. *A polynomial set $\{P_n(x)\}$ is an Appell set if and only*

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[1] C. J. Thorne, *A property of Appell sets*, Amer. Math. Monthly vol. 52 (1945) pp. 191–193.