SOME REMARKS ON THE MEASURABILITY OF CERTAIN SETS

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The present note contains some elementary remarks on sets defined by simple geometric properties. Our main tool will be the Lebesgue density theorem.

First we introduce a few notations: $d(a, b)$ denotes the distance from a to b and $S(x, r)$ the open sphere of center x and radius r. A point x of a set A is said to be of metric density 1 if to every ϵ there exists a δ such that $A \cap S(x, r)$, $r < \delta$, has measure greater than $(1-\epsilon)$ times the volume of $S(x, r)$. $\bar{A}$ denotes the closure of A.

(1) Let E be any closed set in n-dimensional euclidean space. Denote by E_r the set of points whose distance from E is r $(r>0)$. We shall prove that E_r has measure 0.

The set E_r is clearly closed and therefore measurable. If it had positive measure it would contain a point of metric density 1. Let x be any point of E_r and $y \in E$ be one of the points in E at distance r from x. Then $S(y, r)$ cannot contain any point of E_r. Thus x cannot be a point of metric density 1, which completes the proof. This proof is due to T. Radó.

(2) Let A be any set of measure 0 on the positive real axis. Denote by E_A the set of points whose distance from E is in A. We shall show that E_A has measure 0. As is well known A is contained in a G_δ, say G of measure 0. Thus it suffices to show that E_g has measure 0. E_g is clearly a G_δ and thus measurable, so that again it will suffice to show that E_g has no point of metric density 1. Let x be any point of E_g and y any one of the points of E closest to it. Denote by $C_x(\eta_1, \eta_2)$ the half cone defined as follows: $z \in C_x(\eta_1, \eta_2)$ if $d(z, x) < \eta_1$ and the angle zxy is less than η_2. Let R be any ray in C_x from x. Denote by z a variable point of R. We assert that if η_1 and η_2 are sufficiently small, $d(z, E)$ is a decreasing function of $d(z, x)$ for which the upper limit of the difference quotient with respect to $d(z, x)$ is less than $-\delta$, with some $\delta > 0$. Let $y_1 \in E$ be one of the points closest to z in E. We assert that $d(y, y_1)$ is small if η_2 is small. Clearly by definition y_1 is contained in $\{S(z, d(z, y))\}$ but not in $S(x, d(x, y))$. Since $d(x, z) < \eta_1$ the difference of these two spheres has small diameter if η_2 is small, which shows that $d(y, y_1)$ is small. Now it is geometrically clear that for sufficiently small η_1, η_2 there exists a $\delta > 0$ such that the upper limit of the difference quotient of $d(z, y_1)$ with respect to $d(z, x)$ is less

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