THE CONTIGUOUS FUNCTION RELATIONS FOR ${}_pF_q$ WITH APPLICATIONS TO BATEMAN'S $J_n^{u,v}$ AND RICE'S $H_n(\zeta, p, v)$

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1. Introduction. If in the generalized[1] hypergeometric function

$$
\begin{aligned}
{}_pF_q(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; x) &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n} \frac{x^n}{n!}; \\
(\alpha)_n = \alpha(\alpha+1) \cdots (\alpha+n-1), &\ (\alpha)_0 = 1,
\end{aligned}
\tag{1}
$$

one, and only one, of the parameters is increased or decreased by unity, the resultant function is said to be contiguous to the ${}_pF_q$ in (1). We restrict ourselves to the case $p \leqq q+1$, in order to insure a nonzero radius of convergence for the series in (1), and note that no β is permitted to be either zero or a negative integer.

For the ordinary hypergometric function ${}_2F_1$ Gauss[2] obtained fifteen relations each expressing ${}_2F_1$ linearly in terms of two of its six contiguous functions and with coefficients polynomials at most linear in x. Instead of the fifteen relations, it is often convenient to use a set of five linearly independent ones chosen from among them. The other relations all follow from the basic set.

Throughout this study the parameters stay fixed and the work is concerned only with the function ${}_pF_q$ and its contiguous functions. Hence, we are able to use an abbreviated notation illustrated by the following:

$$
\begin{aligned}
F &= {}_pF_q(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; x), \\
F(\alpha_1+) &= {}_pF_q(\alpha_1+1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; x), \\
F(\beta_1-) &= {}_pF_q(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1-1, \beta_2, \cdots, \beta_q; x).
\end{aligned}
$$

There are, of course, $(2p+2q)$ functions contiguous to F. Corresponding to Gauss' five independent relations in the case of ${}_2F_1$ there is for F a set of $(2p+q)$ linearly independent relations, which we shall obtain. The canonical form into which we put this basic set may be described as follows:

First, there are $(p+q-1)$ relations each containing F and two of its contiguous functions. These will be called the simple relations. Each

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[1] For an extensive treatment see W. N. Bailey, *Generalized hypergeometric series*, Cambridge Tract No. 32, 1935.

[2] Gauss, *Werke*, vol. 3, p. 130.