UNION CURVES AND UNION CURVATURE

C. E. SPRINGER

Introduction. A rectilinear congruence in ordinary three-dimensional Euclidean space may be defined by specifying the direction of a unique line at each point of a given surface. A *union curve*[1] on the surface relative to a given congruence has the property that its osculating plane at each point of the curve contains the line of the congruence through the point. It is well known that the union curves relative to the congruence of normals to a surface are the geodesic curves on the surface. The principal aim of this paper is to generalize for union curves certain known results concerning geodesic curves.

The analytical representation of the congruence in §1 is followed in §2 by the derivation of the differential equations of the union curves referred to an arbitrary system of coordinates on the surface. From the definition of the *union curvature vector* in §3, it is seen that a union curve on a surface may be defined as a curve for which the union curvature vector is a null vector at every point of the curve. Finally, there appears in §4 a geometric interpretation of the union curvature of a curve on a surface which agrees with the definition of geodesic curvature of the curve for the particular case of the congruence of normals to the surface.

The notation of Eisenhart[2] will be employed for the most part, although $\Gamma^{\alpha}_{\beta\gamma}$ will be used here as the Christoffel symbol of the second kind. Greek indices will always take the range 1, 2, and Latin indices the range 1, 2, 3. The summation convention of the tensor analysis will be observed.

1. **Analytical representation of the congruence.** Let the surface S be defined analytically with reference to an orthogonal cartesian system of coordinates by

$$x^i = x^i(u^1, u^2) \qquad (i = 1, 2, 3), \tag{1}$$

where the functions x^i and their partial derivatives to the second order are understood to be continuous at any point P on S. Let the line l of the congruence at P have direction cosines given by

$$\lambda^i = \lambda^i(u^1, u^2), \qquad \lambda^i\lambda^i = 1, \tag{2}$$

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[1] Sperry, *Properties of a certain projectively defined two-parameter family of curves on a general surface*, Amer. J. Math. vol. 40 (1918) p. 213.

[2] Eisenhart, *Differential geometry*, Princeton University Press, 1940.