AXIOMATIC CHARACTERIZATION OF FIELDS BY THE PRODUCT FORMULA FOR VALUATIONS

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Introduction. The theorems of class field theory are known to hold for two kinds of fields: algebraic extensions of the rational field and algebraic extensions of a field of functions of one variable over a field of constants. We shall refer to these fields as number fields and function fields, respectively. For class field theory, the function fields must indeed be restricted to those with a Galois field as field of constants; however, we make this restriction only in §5, and until then consider fields with an arbitrary field of constants.

In proving these theorems, the product formula for valuations plays an important rôle. This formula states that, for a suitable set of inequivalent valuations $|\ |_{\mathfrak{p}}$,

$$\prod_{\mathfrak{p}} |\alpha|_{\mathfrak{p}} = 1$$

for all numbers $\alpha \neq 0$ of the field. For fields of the types mentioned, this product formula is easy to prove. After reviewing this proof (§1), we shall show (§2) that, conversely, the number fields and function fields are characterized by their possession of a product formula. Namely, we prove that if a field has a product formula for valuations, and if one of its valuations is of suitable type, then it is either a function field or a number field.

This shows that the theorems of class field theory are consequences of two simple axioms concerning the valuations, and suggests the possibility of deriving these theorems directly from our axioms. We do this in the later sections of this paper for the generalized Dirichlet unit theorem, the theorem that the class number is finite, and certain others fundamental to class field theory. This axiomatic method has the decided advantage of uniting the two cases; also, it simplifies the proofs. For example, we avoid the use of either ideal theory or the Minkowski theory of lattice points. Thus these two theories are unnecessary to class field theory, since they are needed only to prove the unit theorem.

1. **Preliminaries on valuations.** If k is any field, then a function $|\alpha|$, defined for all $\alpha \in k$, is called a valuation of K if:

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