THE EXISTENCE OF ANORMAL CHAINS

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1. Introduction. Let $\bar{B}$ be a Borel field of subsets of a space X, and let $P(x, E)$ be for fixed x a probability measure on $\bar{B}$ and for fixed E a $\bar{B}$-measurable function of x. $P(x, E)$ may be considered as representing the transition probability of going from x into E in a single trial. Denote by Ω the space of sequences $\omega: (x_0, x_1, \cdots)$ where $x_i \in X$ and by $\bar{E}$ the Borel field of subsets of Ω determined by all sets

$$\{x_i \in E\}, \quad \text{where} \quad E \in \bar{B}, \qquad i = 1, 2, \cdots.$$

Doob [2, pp. 102–103][1] has shown that there exists for each $x \in X$ a probability measure $P_x(S)$ defined on $\bar{E}$ such that for every P_x-integrable function $f(x_1, \cdots, x_n)$

$$(1) \quad \int f(\omega) dP_x = \int\int \cdots \int f(x_1, \cdots, x_n) dP(x_{n-1}, x_n) \cdots dP(x, x_1),$$

that Ω with the measure P_x is a Markoff process, that is, $E(x_1, \cdots, x_n; g) = E(x_n; g)$ where $g = g(x_{n+1}, x_{n+2}, \cdots)$ and the E's denote conditional expectations with respect to the indicated variables, and that $E(x_1, \cdots, x_r; f)$ is the function obtained by carrying out the first $n-r$ integrations in (1).

Write $Q(x, E) = P_x(\limsup \{x_i \in E\})$, so that $Q(x, E)$ represents the probability of entering E infinitely often, starting from x. Following Doblin [1, p. 68 et seq.] we make the following definitions for sets of $\bar{B}$: E is *inessential* if $Q(x, E) = 0$ for all x, and *essential* otherwise. An essential set is *improperly essential* if it is a denumerable sum of inessential sets, and *absolutely essential* otherwise. A finite or denumerable sum of improperly essential sets is consequently improperly essential. E is *closed* if $P(x, E) = 1$ for all $x \in E$, and a closed set is indecomposable if it does not contain two disjunct non-empty closed subsets. An absolutely essential indecomposable set is said to be *normal* if it contains a closed set which contains no improperly essential subsets and *anormal* otherwise. If X is a normal set, we shall say that the Markoff chain determined by $P(x, E)$ is a normal chain.

Doblin [1] has obtained for normal chains many elegant results which are considerably more complicated for the anormal case. For example [1, p. 81] in the normal case there exists a closed set G such

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[1] Numbers in brackets refer to the references cited at the end of the paper.