REPRESENTATION OF FOURIER INTEGRALS AS SUMS. I

R. J. DUFFIN

Let $\phi(x)$ be an arbitrary function and let the functions $F(x)$ and $G(x)$ be defined by the series:

$$F\left(x\left(\frac{\pi}{2}\right)^{1/2}\right) = \phi(x) - \frac{1}{3}\phi\left(\frac{x}{3}\right) + \frac{1}{5}\phi\left(\frac{x}{5}\right) - \cdots,$$

$$G\left(x\left(\frac{\pi}{2}\right)^{1/2}\right) = \frac{1}{x}\phi\left(\frac{1}{x}\right) - \frac{1}{x}\phi\left(\frac{3}{x}\right) + \frac{1}{x}\phi\left(\frac{5}{x}\right) - \cdots.$$

Then $G(x)$ is the Fourier sine *transform of $F(x)$; that is,*

$$G(x) = \left(\frac{\pi}{2}\right)^{1/2}\int_0^\infty \sin xtF(t)dt.$$

The purpose of this paper is to give restrictions on $\phi(x)$ so that this relation is valid in some sense.

It will be shown here that the restrictions on $\phi(x)$ are closely related to restrictions which insure the existence and inversion of $\int_0^\infty \sin xt\phi(t)dt$. Well known and important cases for the inversion of the Fourier transform are: 1. $\phi(t)\subset L_2$, 2. $\phi(t)\subset L_1$, and 3. $\phi(t)$ of bounded variation. The analogous cases will be considered.

It is convenient to employ the following notation: $\text{sn } x = \sin(\pi/2)x$, $\text{cs } x = \cos(\pi/2)x$, and $\alpha_n = \sin(\pi/2)n$; $n = 0, 1, 2, \cdots$. Thus we are trying to justify the relation

$$\sum_1^\infty \frac{\alpha_n}{x}\phi\left(\frac{n}{x}\right) = \int_0^\infty \text{sn } xt\sum_1^\infty \frac{\alpha_n}{n}\phi\left(\frac{t}{n}\right)dt.$$

We shall call this in what follows the sine transform. (No confusion should result from the fact that sn x has a different meaning in the theory of elliptic functions.)

The proofs given here do not assume any previous knowledge of Fourier integrals although certain elementary properties of Fourier series are employed.

1. L_2 **theory.** We make the restriction on $\phi(x)$ not only that it belong to L_2 but also that at least one of the series converges suitably to a function in L_2.

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