PROOF OF A THEOREM OF LITTLEWOOD AND PALEY

A. ZYGMUND

1. Introduction. In recent years, important results in the theory of Fourier series were obtained by Littlewood and Paley [3].[1] They used complex methods, and their main tool was an auxiliary function, $g(\theta)$, which they themselves had introduced.

Let $\phi(z)$ be any function regular for $|z|<1$. The real-valued and non-negative function $g(\theta)=g(\theta;\phi)$ is defined by the formula

$$(1.1)\qquad g(\theta) = \left\{\int_0^1 (1-\rho)\,|\phi'(\rho e^{i\theta})|^2 d\rho\right\}^{1/2}, \qquad 0 \leqq \rho < 1.$$

The integral on the right is finite or infinite, but always has meaning.

Let $f(\theta)$ be any L-integrable function of period 2π, and let $f(\rho, \theta)$ be the Poisson integral of f. Thus

$$f(\rho,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(u)P(\rho, \theta-u)du,$$

where $P(\rho, t)=(1-\rho^2)/2(1-2\rho\cos t+\rho^2)$ is the Poisson kernel. If $\bar{f}(\rho,\theta)$ is the harmonic function conjugate to $f(\rho,\theta)$ and vanishing at the origin, and if we set

$$\phi(z) = f(\rho,\theta) + i\bar{f}(\rho,\theta), \qquad z = \rho e^{i\theta},$$

the function (1.1) will sometimes be denoted by $g(\theta;f)$.

The function $g(\theta)$ is suggested by some heuristic argument (see [3, I]). It does not seem to possess any obvious geometric significance, although it has a majorant, $s(\theta)$, with a simple geometric meaning. The reader interested in this problem is referred to papers [4, 7]. In the present note we shall be exclusively concerned with the function $g(\theta)$.

As usual, by H^λ we denote the class of functions $\phi(z)$ regular in $|z|<1$ and satisfying

$$(1.2)\qquad \int_0^{2\pi} |\phi(\rho e^{i\theta})|^\lambda d\theta = O(1), \qquad 0 \leqq \rho < 1.$$

As is well known, this condition implies almost everywhere the existence of the radial limit $\phi(e^{i\theta})=\lim_{\rho\to 1}\phi(\rho e^{i\theta})$.

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[1] Numbers in brackets refer to the references cited at the end of the paper.