NOTE ON THE PRECEDING PAPER

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The sufficiency portion of the theorem on the harmonic series proved by Erdös and Niven in the preceding paper hinges on the fact that (in their notation) $k_2=k$ implies $k_j=k$ for $j>2$. We shall show that this is true more generally for any series $\sum u_n$ such that $\{u_n\}$ is completely monotonic. The result follows at once from the theorem below.

In the case $k_2>k$, the method has thus far not yielded any result of the kind obtained by Erdös and Niven.

THEOREM. *Let $u_n\neq 0$ $(n=1, 2, \cdots)$ be a sequence such that*

$$(1) \qquad (-1)^k\Delta^k u_n \geqq 0 \qquad (k = 0, 1, \cdots ; n = 1, 2, \cdots),$$

that is, $\{u_n\}$ is completely monotonic, and

$$(2) \qquad \lim_{n\to\infty} u_{n+1}/u_n = 1.$$

Define

$$S(n, k) = u_n + u_{n+1} + \cdots + u_{n+k-1},$$
$$f(n, k) = S(n + k, k + 1) - S(n, k).$$

Then $f(n, k)>0$ implies $f(n+1, k)>0$.

We require the following lemma, which is a consequence of a theorem of D. V. Widder.[1]

LEMMA. *Let $\phi(t)$ be a function continuous in $(0, 1)$ and having at most one change of sign in this interval. If $\alpha(t)$ is non-decreasing in $(0, 1)$, then the sequence v_n defined by*

$$v_n = \int_0^1 t^n\phi(t)d\alpha(t), \qquad n = 1, 2, \cdots,$$

has at most one change of sign.

PROOF. If $\phi(t)$ is of constant sign in $(0, 1)$ there is nothing to prove. Suppose then that it changes sign at $t=t_0$. Define $\psi(t)=\int_{t_0}^t \phi(t)d\alpha(t)$. Then $\psi(t)$ has at most one change of trend[2] in $(0, 1)$. Since

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[1] D. V. Widder, *The inversion of the Laplace integral and the related moment problem*, Trans. Amer. Math. Soc. vol. 36 (1934) p. 195.

[2] Loc. cit. p. 155.