ON CERTAIN VARIATIONS OF THE HARMONIC SERIES

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Consider any block of terms from the harmonic series

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+k-1} = S(n, k) \qquad (n \geqq 1, k \geqq 1).$$

Define the integer k_2 by the relation[1]

$$(1) \qquad S(n+k, k_2) < S(n, k) < S(n+k, k_2+1),$$

and similarly $k_3, k_4, \cdots$ by

$$(2) \quad S(n+k+k_2, k_3) < S(n+k, k_2) < S(n+k+k_2, k_3+1),$$

$$(3) \quad \begin{aligned} S(n+k+k_2+k_3, k_4) &< S(n+k+k_2, k_3) \\ &< S(n+k+k_2+k_3, k_4+1), \end{aligned}$$

and so on. We shall study the series

$$(4) \quad \begin{aligned} S(n, k) - S(n+k, k_2) &+ S(n+k+k_2, k_3) \\ &- S(n+k+k_2+k_3, k_4) + \cdots . \end{aligned}$$

THEOREM. *The series* (4) *is convergent if and only if* $k = k_2$.

LEMMA 1. $\log(1+k/n) < S(n, k) < \log(1+2k/(2n-1))$.

The inequality on the left arises from the usual comparison of the harmonic series with the integral of the function $1/x$. To prove the other inequality, we note that the convexity of the function $1/x$ implies

$$\int_{n-1/2}^{n+1/2} \frac{dx}{x} > \frac{1}{n} \quad \text{or} \quad \log \frac{2n+1}{2n-1} > \frac{1}{n} \cdot$$

We replace n by $n+1, n+2, \cdots, n+k-1$ in the latter inequality and add the results.

LEMMA 2. *If* $k = k_2$, *then* $k = k_j$ *with* $j > 2$, *and the series* (4) *converges.*

We need prove only that $k = k_3$. Since $k > k_3$ is not possible, let us

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[1] It is not possible that $S(n, k) = S(n+k, k_2)$. For let h be the unique integer in the range $(n, n+k+k_2-1)$ which is divisible by the highest power of 2, say 2^r. Let m be the l.c.m. of all the odd divisors of $n, n+1, \cdots, n+k+k_2-1$. Then $2^{r-1}mS(n, k) = 2^{r-1}mS(n+k, k_2)$ is an equation involving $k+k_2-1$ integers and the one fraction $m2^{r-1}/h$.