ON TOPOLOGIES FOR FUNCTION SPACES

RALPH H. FOX

Given topological spaces X, T, and Y and a function h from $X \times T$ to Y which is continuous in x for each fixed t, there is associated with h a function h^* from T to $F = Y^X$, the space whose elements are the continuous functions from X to Y. The function h^* is defined as follows: $h^*(t) = h_t$, where $h_t(x) = h(x, t)$ for every x in X. The correspondence between h and h^* is obviously one-to-one.

Although the continuity of any particular h depends only on the given topological spaces X, T, and Y, the topology of the function space F is involved in the continuity of h^*. It would be desirable to so topologize F that the functions h^* which are continuous are precisely those which correspond to continuous functions h. It has been known for a long time that this is possible if X satisfies certain conditions, chief among which is the condition of *local compactness* (Theorem 1). This condition is often felt to be too restrictive (since it practically excludes the possibility of X itself being a function space), and several years ago, in a letter, Hurewicz proposed to me the problem of defining such a topology for F when X is not locally compact. At that time I showed by an example (essentially Theorem 3) that this is not generally possible. Recently I discovered that, by restricting the range of T in a very reasonable way, one of the standard topologies for F has the desired property even for spaces X which are not locally compact (Theorem 2). In this last result the condition of local compactness is replaced by the first countability axiom and this appeals to me as a less troublesome condition.

It should be pointed out that the problem is motivated by the special case in which T is the unit interval. When T is the unit interval, h is a homotopy and h^* is a path in the function space; in the topology of deformations, equivalence of the concepts of "homotopy" and of "function-space path" is usually required.

Among the various possible topologies for F there is one, which I shall call the compact-open[1] (co.o.) topology, which seems to be the most natural. For any two sets, A in X and W in Y, let $M(A, W)$ denote the set of mappings $f \in F$ for which $f(A) \subset W$. The co.o. topology is defined by selecting as a sub-basis for the open sets of F the

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[1] Terminology followed in this note is generally that of Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloquium Publications, vol. 27, New York, 1942.