THE MANIFOLDS OF LINEAR ELEMENTS OF AN n-SPHERE

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1. Introduction. The 3-manifolds of oriented and non-oriented linear elements of closed surfaces have been investigated by Nielsen,[1] Hotelling,[2] Threlfall,[3] van der Waerden and others.[4] In the present paper we take up the case of the space M of oriented linear elements, and the space M' of non-oriented linear elements, of an n-sphere, $n \geqq 1$. The chief tools in the present investigation are certain orthogonal transformations (§§3–4) and theorems on addition of complexes.[5] Our success in the determination of certain homology classes (§§7–8, 14) leads to complete determination of (integral) Betti groups of M and M'. Our results may be summarized as follows:

(M1) For $n>1$, M is an orientable $(2n-1)$-manifold. Its Betti groups, which are not the null groups, are the following: For even n, B^0 and $B^{2n-1} \approx G_0$ (AH, p. 556) and $B^{n-1} \approx G_2$; for odd n, B^0, B^{2n-1}, B^{n-1}, and $B^n \approx G_0$.

(M2) For $n=2$, M is the projective space. For $n>2$, its fundamental group is the identity.

(M3) For $n=1, 3, 7$, M is the topological product of an n-sphere and an $(n-1)$-sphere.

(M'1) For $n>1$, M' is an orientable or a non-orientable $(2n-1)$-manifold according as n is even or odd. Its Betti groups, which are not the null, are the following: For even n, B^0 and $B^{2n-1} \approx G_0$, $B^{n-1} \approx G_4$, and $B^r \approx G_2$, $r=1, 3, \cdots, n-3; n+1, n+3, \cdots, 2n-3$. For odd n, B^0 and $B^n \approx G_0$, and $B^r \approx G_2$, $r=1, 3, \cdots, n-2; n+1, n+3, \cdots, 2n-2$.

(M'2) For $n=2$, M' is the lens space (Linsenraum) (4,1).[6] For $n>2$, its fundamental group is the cyclic group of order 2.

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[1] J. Nielsen, *Untersuchungen zur Topologie der geschlossen zweiseitigen Flächen*, Acta. Math. vol. 50 (1927) pp. 302–306.

[2] H. Hotelling, *Three-dimensional manifolds of states of motions*, Trans. Amer. Math. Soc. vol. 27 (1925) pp. 329–344; *Multiple-sheeted spaces and manifolds of states of motions*, ibid. vol. 28 (1926) pp. 479–490.

[3] W. Threlfall, *Räume aus Linienelementen*, Jber. Deutschen Math. Verein. vol. 42 (1933) I, pp. 88–110.

[4] *Solutions of problem* 124 by B. L. van der Waerden, H. Kneser, H. Seifert, E. R. van Kampen, and W. Threlfall, Jber. Deutschen Math. Verein. vol. 42 (1933) II, pp. 112–117.

[5] Alexandroff-Hopf, *Topologie* I, Berlin (1935), pp. 287–293. This book will be referred to as AH.

[6] Seifert-Threlfall, *Lehrbuch der Topologie*, Leipzig, 1934, p. 210. This book will be referred to as ST.