A NOTE ON HYPERGEODESICS AND CANONICAL LINES

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In this note we introduce two families of hypergeodesics on a non-ruled surface in ordinary projective space. Consideration of the properties of these hypergeodesics leads to certain geometrical constructions which yield canonical lines of the first kind from a given canonical line of the second kind.

We shall assume that the differential equations of a non-ruled surface S are written in the Fubini canonical form[1]

$$(1)\quad x_{uu} = px + \theta_u x_u + \beta x_v, \qquad x_{vv} = qx + \gamma x_u + \theta_v x_v \quad (\theta = \log \beta\gamma).$$

We select an ordinary point P_x of the surface S as one vertex of the usual local tetrahedron of reference. When a curve C_λ through the point P_x is regarded as being imbedded in the one-parameter family of curves represented on S by the equation

$$(2)\qquad dv - \lambda(u, v)du = 0,$$

the osculating plane at the point P_x of the curve C_λ has the local equation

$$(3)\qquad 2\lambda(\lambda x_2 - x_3) + (\lambda' + \beta - \theta_u\lambda + \theta_v\lambda^2 - \gamma\lambda^3)x_4 = 0,$$

in which we have placed $\lambda' = \lambda_u + \lambda\lambda_v$.

It will be recalled that two lines $l_1(a, b)$, $l_2(a, b)$ are reciprocal lines[2] at a point P_x of a surface if the line $l_1(a, b)$ joins the point P_x and the point y defined by placing

$$y = -ax_u - bx_v + x_{uv}$$

and the line $l_2(a, b)$ joins the points ρ, σ defined by

$$\rho = x_u - bx, \qquad \sigma = x_v - ax,$$

where a, b are functions of u, v. As the point P_x varies over the surface S, the lines $l_1(a, b)$, $l_2(a, b)$ generate two reciprocal congruences Γ_1, Γ_2, respectively.

The two reciprocal lines $l_1(a, b)$, $l_2(a, b)$ are canonical lines $l_1(k)$, $l_2(k)$ of the first and second kind respectively in case

$$a = -k\psi, \qquad b = -k\phi,$$

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[1] E. P. Lane, *Projective differential geometry of curves and surfaces*, Chicago, 1932, p. 69.

[2] E. P. Lane, *loc. cit.*, pp. 82–85.