ALGEBRAIC DETERMINATION OF THE SECOND FUNDAMENTAL FORM OF A SURFACE BY ITS MEAN CURVATURE

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1. Introduction. The main purpose of the following paper is to show that in general the mean curvature of a surface yields an *algebraic* determination of its second fundamental form. We do this by deriving the explicit equations giving this determination. The continuity and differentiability properties of the various functions entering into the discussion will be assumed without special mention since these requirements are obvious from the methods and equations employed.

We denote the mean curvature by H and the Gaussian curvature by K. The symbols $g_{\alpha\beta}$ and $b_{\alpha\beta}$ will be used to denote the symmetric components of the first and second fundamental forms of the surface (two-dimensional surface in Euclidean three space). Between these quantities we have the relations

$$(1.1) \qquad H = g^{\alpha\beta} b_{\alpha\beta}/2 \quad \text{and} \quad |g_{\alpha\beta}| K = |b_{\alpha\beta}| = b_{11}b_{22} - b_{12}^2,$$

where $|g_{\alpha\beta}|$ and $|b_{\alpha\beta}|$ stand for determinants. The first of these relations can be regarded as defining the mean curvature. The second is known as the Gauss equation. We may mention here also the Codazzi equations which play an important role in the following, that is, the equations $b_{\alpha\beta,\gamma} = b_{\alpha\gamma,\beta}$, where the "comma" denotes covariant differentiation based on the first fundamental form of the surface.

We shall find that the combination $H^2 - K$ enters into most of the following equations. This quantity satisfies the condition $H^2 - K \geqq 0$. For, if we choose a coordinate system such that at a point P we have $g_{\alpha\beta} = \delta_{\alpha\beta}$, then at this point $2H = b_{11} + b_{22}$ and hence

$$\begin{aligned} 4(H^2 - K) &= (b_{11}^2 + 2b_{11}b_{22} + b_{22}^2) - 4(b_{11}b_{22} - b_{12}^2) \\ &= (b_{11}^2 - 2b_{11}b_{22} + b_{22}^2) + 4b_{12}^2 \\ &= (b_{11} - b_{22})^2 + 4b_{12}^2 \geqq 0. \end{aligned}$$

Suppose that $H^2 - K = 0$ in a region R of the surface. Then at a point P of R and relative to a coordinate system for which $g_{\alpha\beta} = \delta_{\alpha\beta}$ at P we have

$$(1.2) \qquad b_{11} + b_{22} = 2H \quad \text{and} \quad b_{11}b_{22} - b_{12}^2 = H^2.$$

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