# A NEW APPLICATION OF THE SCHUR DERIVATE

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Fermat's theorem in elementary number theory states that if $p$ is a rational prime, $a$ an integer,

$$a^p \equiv a \pmod{p}.$$

Hence

$$a^{p^{n+1}} \equiv a^{p^n} \pmod{p^{n+1}}$$

or

$$(a^{p^{n+1}} - a^{p^n})/p^{n+1} \tag{0, 1}$$

is a rational, hence a $p$-adic, integer.

By introducing as the derivate, $\Delta a_n$, of a sequence $\{a_n\}$ with respect to the number $p$ the expression

$$\Delta a_n = (a_{n+1} - a_n)/p^{n+1}, \tag{0, 2}$$

I. Schur[1] in 1933 generalized Fermat's theorem. The Fermat theorem states that the first Schur derivate of the sequence $\{a^q\}$ with $q=p^n$, (0, 1), is integral. Schur proved the generalization that, if $a$ is prime to $p$, not only the first derivate, but the higher Schur derivates up to the $(p-1)$st are integral (in the $p$-adic or rational sense). Zorn[2] in 1936 extended this result by proving that all Schur derivates of $\{a^q\}$ with $q=p^n$ are $p$-adically bounded, hence convergent, and discussing the $p$-adic function, $\lim_{n\to\infty} \Delta^m a^q$, where $q=p^n$.

It is a fact[3] of elementary number theory that the sum of the $k$th ($k$ a positive or negative integer or zero) powers of the rational integers less than and prime to $p^n$ ($p$ a rational prime, $n$ a positive integer) is divisible by $p^n$ if $p-1$ does not divide $k$ or by $p^{n-1}$ if $p-1$ divides $k$. The quotient of the division of such a sum by $p^n$,

$$S[n, x^k] = \sum_{i=1}^{q}{}' i^k/p^n, \tag{0, 3}$$

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[1] Preuss. Akad. Wiss. Sitzungsber. (1933) p. 145.

[2] Ann. of Math. vol. 38 (1937) pp. 451–464.

[3] For a recent proof see H. Gupta, Proceedings of the Indian Academy of Sciences, Section A, vol. 13 (1944) pp. 85–86. His theorem is stated for even $k$, but the evenness of $k$ is not used. Note that all concepts used are defined for negative $k$ and the same proof holds. Classic results in number theory are Wolstenholme's theorem and Leudesdorf's generalization which yield divisibility by $p^{2n}$ for $k=-1$, $p>3$.