A NOTE ON THE REPLICAS OF NILPOTENT MATRICES

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In a recent paper,[1] Chevalley proved the following theorem:

(A) *If Z is a nilpotent matrix over a field K of characteristic* 0, *the only replicas Z' of Z are the matrices $Z'=tZ$, $t\in K$.*[2]

For the proof of (A), he made use of a particular case of a theorem due to Ado and gave a proof for the results which he needed. In the present note, we shall give a direct simple proof of (A) and we shall in fact deduce it as an immediate consequence of the stronger theorem:

(B) *If Z and Z' are two nilpotent matrices over a field K of characteristic* 0, *and if $q(x)$ and $r(x)$ are two polynomials with coefficients in K and without constant terms such that $Z'=q(Z)$ and $Z'_{0,2}=r(Z_{0,2})$, then $Z'=tZ$, $t\in K$.*

We shall later establish corresponding results for fields K of prime characteristics, to be stated as theorems (C) and (D).

That (A) is implied by (B) follows immediately from the fact that if Z' is a replica of Z, then $Z'_{r,s}=p_{r,s}(Z_{r,s})$, where $p_{r,s}(x)$ are polynomials in K without constant terms.[3]

For the proof of (B), let n be the degree of Z and Z' and let m be the least nonnegative integer such that $Z^{m+1}=0$. Clearly $0\leqq m\leqq n-1$. The case $Z=0$ is trivial; we can therefore assume $1\leqq m\leqq n-1$. Let also l be the least nonnegative integer such that $(Z_{0,2})^{l+1}=0$. Clearly $Z_{0,2}$ is nilpotent and $1\leqq l\leqq n^2-1$. We shall see that $m\leqq l\leqq 2m$.

The matrix Z can be transformed by an (n, n) matrix T with coefficients in the algebraic closure $\overline{K}$ of K into the following form:

$$Z_1 = T^{-1}ZT = \begin{pmatrix} 0 & & & & \\ z_1 & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ & & & z_{n-1} & 0 \end{pmatrix}, \tag{1}$$

where $z_1, \cdots, z_{n-1}$ are zeros and ones and not all zeros. Then for

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[1] Claude Chevalley, *On a kind of new relationship between matrices*, Amer. J. Math. vol. 65 (1943) pp. 521–531.

[2] Theorem 6, p. 530, loc. cit.

[3] Lemma 4, p. 529, loc. cit.