NONCOMMUTATIVE VALUATIONS

O. F. G. SCHILLING

The topic of this paper is the extension of the basic facts of valuation theory to noncommutative systems.[1] The purpose of this generalization is twofold. First, the theory of valuations with commutative groups of values is placed in the framework of the theory of l-groups,[2] and secondly the general theory leads to the construction of a new class of infinite division algebras. These division algebras are of highly transcendental structure over their respective centers; moreover they may be considered, in special cases, as crossed transcendental extensions of other division algebras.

It is necessary to recall some facts on l-groups. A group Γ is called a simply ordered l-group if the following axioms are satisfied:

(I) There is defined a binary inclusion relation which is "homogeneous" in the sense that $\alpha \geqq \beta$ implies $\rho+\alpha+\sigma \geqq \beta+\beta+\sigma$ for all ρ, σ,

(II) Γ is a lattice with respect to the ordering relation, and

(III) given α, β, either $\alpha \geqq \beta$ or $\beta \geqq \alpha$.[3]

Furthermore $\alpha \geqq \beta$ means $\alpha \cup \beta = \alpha$. The totality of all positive elements of Γ is a semi-group and shall be denoted by Γ^+. The absolute value $|\alpha|$ of α is defined as $\alpha \cup -\alpha$. Hence $|\alpha|$ is equal to α or $-\alpha$ according as α lies in Γ^+ or the complement $\Gamma - \Gamma^+$.[4] Since Γ is simply ordered an l-ideal or isolated subgroup Δ which is defined by Birkhoff[5] to contain with each δ all ξ with $|\xi| < |\delta|$ may alternately be defined as follows. An isolated subgroup contains with each $\delta > 0$ all $\xi \in \Gamma^+$ satisfying $\xi < \delta$.

DEFINITION. *A one-valued function V on a division ring D upon an l-group Γ is called a valuation if the following postulates hold*:

Presented to the Society, August 14, 1944; received by the editors May 29, 1944.

[1] For results on valuation theory see, for example, W. Krull, *Allgemeine Bewertungstheorie*, J. Reine Angew. Math. vol. 167 (1932) pp. 160–196; O. F. G. Schilling, *Arithmetic in fields of formal power series in several variables*, Ann. of Math. vol. 38 (1937) pp. 551–576; Saunders MacLane, *The uniqueness of the power series representation of certain fields with valuations*, Ann. of Math. vol. 39 (1938) pp. 370–382; A. Ostrowski, *Untersuchungen zur arithmetischen Theorie der Körper*, Math. Zeit. vol. 39 (1935) pp. 269–404; I. Kaplansky, *Maximal fields with valuations*, Duke Math. J. vol. 9 (1942) pp. 303–321.

[2] Garrett Birkhoff, *Lattice-ordered groups*, Ann. of Math. vol. 43 (1942) pp. 298–331.

[3] Birkhoff, loc. cit. pp. 299, 300, 312.

[4] Birkhoff, loc. cit. pp. 302, 308, 309.

[5] Birkhoff, loc. cit. pp. 309–311.