ON APPROXIMATE ISOMETRIES

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In a previous paper, a problem of mathematical "stability" for the case of the linear functional equation was studied.[1] It was shown that if a transformation $f(x)$ of a vector space E_1 into a Banach space E_2 satisfies the inequality $\|f(x+y)-f(x)-f(y)\|<\epsilon$ for some $\epsilon>0$ and all x and y in E_1, then there exists an additive transformation $\phi(x)$ of E_1 into E_2 such that $\|f(x)-\phi(x)\|<\epsilon$.

In the present paper we consider a stability problem for isometries. By an ϵ-isometry of one metric space E into another E' is meant a transformation $T(x)$ which changes distances by at most ϵ, where ϵ is some positive number; that is, $|\rho(x, y)-\rho(T(x), T(y))|<\epsilon$ for all x and y in E. Given an ϵ-isometry $T(x)$, our object is to establish the existence of a true isometry $U(x)$ which approximates $T(x)$; more precisely, to establish the existence of a constant $k>0$ depending only on the metric spaces E and E' such that $\rho(T(x), U(x))<k\epsilon$ for all x in E. In this paper this result will be proved for the case in which $E=E'$, where E is n-dimensional Euclidean space or Hilbert space (not necessarily separable). The case in which E is the space C of continuous functions will be treated in another paper.

The above problem of ϵ-isometries is related to the problem of constructing space models for sets in which distances between points are given only with a certain degree of exactness (measurements are possible only with a certain degree of precision). The question of the uniqueness of the idealized model corresponding to the given measurements and the extrapolation from the measurements to the model could be looked upon as a problem in determining a strict isometry from an approximate isometry.

In the case of certain simple metric spaces, for example the surface of the Euclidean sphere, this question can be answered in the affirmative, but it may be more difficult for other bounded manifolds. A simple but interesting example showing a case where the answer is *negative* has been worked out by R. Swain.

THEOREM 1. *Let E be a complete abstract Euclidean vector space.*[2]

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[1] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A. vol. 27 (1941) pp. 222–224.

[2] A complete Euclidean vector space is a Banach space whose norm is generated by an inner product, (x, y). It includes real Hilbert space and n-dimensional Euclidean spaces as special cases.