THE CONVERGENCE AND VALUES OF PERIODIC CONTINUED FRACTIONS

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1. Introduction. Much of the recent progress in the study of continued fractions is due to the consideration of the continued fraction as a product of linear fractional transformations, as opposed to the older approach in which attention is focused on the numerators and denominators, A_n and B_n, of the approximants.

The use of the transformation point of view, besides simplifying the proofs of theorems, often sheds additional light on the significance of the results. This advantage is particularly noticeable in the case of the periodic continued fraction, as the following theorem and its proof will show. The essential portions of the theorem were given by Stolz [1].[1] An improved but lengthy proof was later given by Pringsheim [2]. This was followed by Perron's shorter proof [3], in which ideas related to the transformation point of view were virtually suppressed. The proof given here makes full use of linear fractional transformations and is even shorter than Perron's proof.

THEOREM. *Consider the k-term periodic continued fraction*

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_k}{b_k} + \frac{a_1}{b_1} + \cdots, \tag{1.1}$$

for which $a_1a_2 \cdots a_k \neq 0$. *Let* F_ν *denote the* ν*th approximant, so that* $F_\nu = A_\nu/B_\nu$. *Let S denote the linear fractional transformation*

$$z' = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_k}{b_k} + \frac{z}{1}, \tag{1.2}$$

and let x_1 *and* x_2 *be the fixed points of S. The continued fraction converges if and only if* x_1 *and* x_2 *are finite numbers satisfying one of the following two conditions:*

$$x_1 = x_2 \tag{1.3}$$

or

$$F_r \neq x_2,\ r = 0, 1, 2, \cdots, k-1, \text{ and } |F_{k-1} - x_1| < |F_{k-1} - x_2|. \tag{1.4}$$

If the continued fraction converges, its value is x_1. *Furthermore,* $F_{nk+r} = x_1$ *for some r,* $0 \leqq r \leqq k-1$, *and for* $n = 1, 2, 3, \cdots$, *if and only if* $F_r = x_1$.

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.