CONCERNING THE DEFINITION OF HARMONIC FUNCTIONS

E. F. BECKENBACH

1. **Introduction.** A real function $u(x, y)$, defined in a domain (non-null connected open set) D, is said to be harmonic in D provided $u(x, y)$ and its partial derivatives of the first and second orders are continuous and the Laplace equation,

$$\Delta u \equiv \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0, \tag{1}$$

is satisfied throughout D. A function is said to be harmonic at a point provided it is harmonic in a domain containing the point.

It has been shown [1][1] that if $u(x, y)$ is continuous in D and if the second order partial derivatives $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ exist and satisfy the Laplace equation (1) throughout D, then $u(x, y)$ is harmonic in D.

We shall show that if $u(x, y)$ and its partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ are continuous in D, if $\partial u/\partial x$ and $\partial u/\partial y$ are differentiable, or even have finite Dini derivates, with respect to x and y at all points of D except at most at the points of a denumerable set of points in D, and if the Laplace equation (1) is satisfied at almost all points of D at which $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ exist, then $u(x, y)$ is harmonic in D.

Our result is comparable with the Looman-Menchoff theorem [**3**, pp. 9–16; **5**, pp. 198–201] concerning the Cauchy-Riemann first order partial differential equations and analytic functions of a complex variable. Ridder [**4**] has stated that harmonic functions can be given a Looman-Menchoff characterization; but a generalization of the Looman-Menchoff theorem on which his proof is based is invalid, for there are functions having isolated singularities which satisfy the hypotheses of the generalization without satisfying the conclusion. For a generalization of the Looman-Menchoff theorem, see Maker [**2**].

2. **Notation and lemmas.** By $C(Q)$ we shall denote a square, by $C(R)$ a rectangle, having sides parallel to the coordinate axes. The set consisting of the points of $C(Q)$, or of $C(R)$, plus its interior, will be denoted by Q, or R, respectively.

Let F be a non-null set closed with respect to the domain D, and $C(Q)$ any square with Q lying in D, with sides of positive length and parallel to the coordinate axes, and with center at a point of F. Then the points common to F and Q will be called a *portion* of F.

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[1] Numbers in brackets refer to the references cited at the end of the paper.