SPANS IN LEBESGUE AND UNIFORM SPACES OF TRANSLATIONS OF STEP FUNCTIONS

RALPH PALMER AGNEW

1. **Introduction.** For each $p \geqq 1$, let L_p be the Lebesgue space whose elements are real or complex valued measurable functions $f(x)$, defined over $-\infty < x < \infty$, for which the integral

$$(1.1) \qquad \int_{-\infty}^{\infty} | f(x) |^p dx$$

is finite. The distance $\|f_2 - f_1\|$ between two elements f_1 and f_2 of the space is defined by

$$(1.2) \qquad \| f_2 - f_1 \| = \left\{ \int_{-\infty}^{\infty} | f_2(x) - f_1(x) |^p dx \right\}^{1/p}.$$

For each $p \geqq 1$, L_p is a linear metric complete separable space.

Let E be a set in L_p. The *linear manifold* $M(E)$ determined by E is the set of all linear combinations (finite) of elements of E. The *span* $S_p(E)$ of E in L_p is the closure in L_p of $M(E)$; an element ϕ of L_p belongs to $S_p(E)$ if and only if to each $\epsilon > 0$ corresponds an element f_ϵ of $M(E)$ such that $\|\phi - f_\epsilon\| < \epsilon$.

Let $f \in L_p$. For each real h, the *translation* $f(x+h)$ of $f(x)$ is also in L_p. Let T_f denote the set of translations of f. Wiener [**2**, pp. 7–9][1] showed that if $f \in L_2$, then $S_2(T_f)$ is the whole space L_2 if and only if the real zeros of the Fourier transform of f form a set of measure 0. He [**2**, pp. 9–19] showed also (and this was much more difficult) that if $f \in L_1$, then $S_1(T_f)$ is the whole space L_1 if and only if the Fourier transform of f has no real zeros. He [**2**, p. 93] raised the question whether similar propositions hold for other values of p and expressed a "suspicion" that they do, at least when $1 \leqq p \leqq 2$.

In view of the similar suspicions held by Wiener and others, a result recently announced by Segal [**1**] is surprising. Segal has shown that if $1 < p < 2$, then there is an element f of L_p such that (i) the zeros of the Fourier transform of f form a set of measure 0 and (ii) the span $S_p(T_f)$ of the translations of f does *not* include the whole space L_p.

This development will doubtless create interest in the search for criteria for $S_p(T_f) = L_p$. With the hope that both the result and the

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.