ON THE COMPOSITION OF ALGEBRAIC FORMS OF HIGHER DEGREE

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1. **Introduction.** No discussion of composition can properly be introduced except by mentioning the complex numbers. From the relation

$$x_1 + ix_2 = (y_1 + iy_2)(z_1 + iz_2),$$

$$x_1 = y_1z_1 - y_2z_2, \qquad x_2 = y_1z_2 + y_2z_1,$$

we have, upon taking norms,

$$x_1^2 + x_2^2 = (y_1^2 + y_2^2)(z_1^2 + z_2^2).$$

Thus the quadratic form $y_1^2+y_2^2$ is composed with itself.

Similarly by using quaternions we achieve the composition of a sum of four squares with itself,[1] and by using Cayley's algebra of order 8, we obtain the composition of a sum of eight squares with itself. No further progress in this direction is possible.[2]

In general let us assume that

$$x_k = \sum_{i,j} c_{ijk}y_iz_j \qquad (i, j, k = 1, 2, \cdots, n) \tag{1}$$

where the c_{ijk} are numbers of a commutative ring $\mathfrak{R}$ with unit element, and the y's and z's are indeterminates. If there exist three homogeneous forms f, g, h of degree k with coefficients in $\mathfrak{R}$ such that

$$f(x) = g(y)\cdot h(z) \tag{2}$$

is an identity by virtue of the bilinear transformation (1), we say that g and h are *composable*, that f is *composite*, and that the transformation (1) is *admissible*.

The theory of composition appears in the mathematical literature under two rather distinct special cases, first where $\mathfrak{R}$ is a field, and second where $\mathfrak{R}$ is the ring of rational integers. The field case is the simpler and progress has been carried much further, but this is not the case in which we are here primarily interested.

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[1] It seems certain that a knowledge of the fact that a sum of four squares is composable with itself, which had been proved by Euler in 1748, was one of the main clues which led Hamilton to the discovery of quaternions.

[2] See L. E. Dickson, Ann. of Math. (2) vol. 20 (1919) pp. 155–171.