A DIFFERENTIAL INEQUALITY

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The following theorem was discovered by S. B. Jackson in connection with a problem in differential geometry.

THEOREM 1. *If $f(x)$ is of class C^2 in $(0, a)$, $f(0)=f'(0)=0$, and $f''(x) \leqq K|f'(x)| + L|f(x)|$ in $(0, a)$, where K and L are constants, then $f(x) \leqq 0$ in some interval $(0, b)$.*

If $f(x)$ is in addition analytic at 0, Theorem 1 becomes trivial, since if a_n is the first nonvanishing coefficient of the power series of $f(x)$, $x^{2-n}f''(x)$ approaches a nonzero limit, while $x^{2-n}f'(x)$ and $x^{2-n}f(x)$ approach zero, as $x \to 0$, and consequently $a_n < 0$.

I shall prove the following more general theorem.

THEOREM 2. *If $f'(x)$ is absolutely continuous in $(0, a)$, $f(0)=f'(0)=0$, and*

$$f''(x) \leqq K(x)\,|f'(x)| + x^{-1}L(x)\,|f(x)| \tag{1}$$

almost everywhere in $(0, a)$, where $K(x)$ and $L(x)$ are non-negative and integrable in $(0, a)$, then either $f(x) \equiv 0$ in some interval $(0, b)$, or $f'(x) < 0$ in $0 < x < \min(a, c)$, where c is such that

$$\int_0^x \{K(t) + L(t)\}\,dt < 1, \qquad 0 < x < c. \tag{2}$$

Since $f(0)=0$, $f(x)$ is negative in $(0, c)$ when $f'(x) < 0$ in $(0, c)$. Theorem 1 is contained in the special case $K(t)=K$, $L(t)=Lt$.

Theorem 2 is the best possible result of its kind; for, if $\int_0^x K(t)dt$ diverges and $K(t)$ is positive and continuous in $t > 0$, the function $f(x)$ defined by $f'(x) = 1/\int_x^1 K(t)dt$, $f'(0)=f(0)=0$, is positive in $x > 0$ and satisfies (1) for all x such that $\int_x^1 K(t)dt > 1$.

Assume that $f(x)$ is not identically zero in any interval $(0, b)$, and write $M(x) = \max_{0 \leqq t \leqq x}|f'(t)|$, so that $M(x) > 0$ for $x > 0$ and $M(x)$ is nondecreasing.

We observe that for $0 \leqq x \leqq a$,

$$|f(x)| = \left| \int_0^x f'(t)dt \right| \leqq xM(x). \tag{3}$$

There are points x_n such that $x_n \downarrow 0$ and $M(x_n) = |f'(x_n)|$. Suppose that $f'(x_n) > 0$ for some n. Let (a_n, b_n) be the largest interval, contain-

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