SYMBOLIC SOLUTION OF CERTAIN PROBLEMS IN PERMUTATIONS

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1. Introduction. It is our purpose in this paper to show how algebraic symbolism can be applied to the problem of enumerating certain types of restricted permutations. The method rests on a symbolic interpretation of the method of inclusion and exclusion, or rather its probability analogue, the formula of Poincaré; the use of the terminology of probability is a matter of convenience only, the considerations being purely combinatorial. The problems considered are of two kinds: those where the restriction of position of elements is absolute (§3, "card-matching" problems), and those where elements are restricted relative to others (§§5, 6). In §4 approximations to the symbolic formulas are obtained. Though these problems are rather special in character, the foundations of the method (§2) are quite general and should find other applications.

2. Symbolic expressions. Consider n events $A_1, \cdots, A_n$ and let $p(A_{i_1} \cdots A_{i_k})$ denote the probability of the joint occurrence of $A_{i_1} \cdots A_{i_k}$. The probability that none of $A_1, \cdots, A_n$ occurs, which we shall denote by P_0, is given by Poincaré's formula:

$$P_0 = 1 - \sum p(A_i) + \sum p(A_i A_j) \cdots . \tag{1}$$

In the case of complete symmetry, that is, where each $P(A_{i_1} \cdots A_{i_k})$ is a function ϕ_k of k alone,

$$P_0 = 1 - n\phi_1 + {}_nC_2\phi_2 - \cdots .$$

By using the displacement operator E defined by $E^k\phi_0 = \phi_k$, we may write more compactly $P_0 = (1-E)^n\phi_0$.

In the cases which we shall consider, however, this complete symmetry will be lacking and instead we shall have the following partial substitute, which we shall call *quasi-symmetry*: $p(A_{i_1} \cdots A_{i_k})$ is either equal to zero or to a function ϕ_k of k alone. To evaluate P_0 for such cases a symbolic device of Broderick [2][1] is helpful; we may write (1) in the form

$$P_0 = (1 - pA_1)(1 - pA_2) \cdots (1 - pA_n), \tag{2}$$

where the multiplication is symbolic in the sense that $p(A_i)p(A_j) \cdots$

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.