LIMITS FOR THE CHARACTERISTIC ROOTS OF A MATRIX[1]

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Let A be a square matrix of order n with complex numbers as elements. The equation $|\lambda I - A| = 0$ is called the characteristic equation of the matrix A, and the roots λ_i, the characteristic roots of the matrix A. Although it is not possible to make any definite statements regarding the nature of the characteristic roots for the general matrix, several authors have given upper limits to the roots. In 1900, Bendixson [1][2] obtained upper limits for the real and imaginary parts of the characteristic roots of a real matrix. In 1902, Hirsch [5] extended these results to matrices with complex numbers as elements. A limit was also given by Bromwich [2] in 1904. These limits were further refined by Browne [3] in 1930, and by Parker [7] in 1937.

In 1918, Toeplitz [8], using the results of Bendixson and Hirsch, studied the algebraic form $(Ax|x) \equiv \sum_{r,s} a_{rs} x_s \bar{x}_r$ corresponding to the matrix A, where by hypothesis the algebraic form $(x|x) = \sum_{r=1}^{n} x_r \bar{x}_r$ has the value unity. By using the fact that A may be decomposed uniquely in the form $A = B + iC$, where B and C are Hermitian, he showed that the totality of values which the algebraic form assumes lie within a rectangle with sides parallel to the real and imaginary axes. He further showed that this field of values is bounded by a convex curve. Hausdorff [4] showed that the field of values is connected, bounded, closed, and convex. In 1932, Murnaghan [6], using the fact that $(Ax|x)$ has values invariant under unitary transformations of A, showed that for normal matrices (that is, matrices which can be transformed unitarily into diagonal form) the field of values is a convex polygon. For non-normal matrices, he showed that in the general case the characteristic roots of the matrix are the foci of the curve bounding the field of values. Wintner had previously remarked that for $n = 2$ the curve is an ellipse. The first three theorems below give radii of circles within which not merely the characteristic roots but also the entire field of values lies.

Let A' and $\bar{A}$ denote the transpose and conjugate, respectively, of the matrix A, and write

$$B = (A + \bar{A}')/2, \qquad C = (A - \bar{A}')/2i.$$

Received by the editors March 10, 1944, and, in revised form, April 10, 1944.

[1] Partially extracted from a dissertation, University of California, 1944.

[2] Numbers in brackets refer to the Bibliography at the end of the paper.