AN APPLICATION OF LATTICE THEORY TO QUASIGROUPS

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The purpose of this note is to show that O. Ore's general formulation of the Jordan-Hölder theorems in partially ordered sets[1] [3] yields the Jordan-Hölder theorems for loops [1] which have recently been obtained by A. A. Albert [2]. We shall assume that the reader is familiar with the papers [1, 2, 3] of O. Ore and of A. A. Albert.

We begin with an examination of certain congruence relations in loops. We are led to characterize the normal divisors of A. A. Albert as those subloops which commute and associate with the elements of the loop. Our proof of the fundamental quadrilateral condition of O. Ore [3] is then based on this characterization.

A *loop* $\mathfrak{G}$ is a quasigroup with an identity element e. For each non-null subset $\mathfrak{H} \subset \mathfrak{G}$ we define a relation H_ρ on $\mathfrak{G}\mathfrak{G}$ as follows:

(1) *If* $x, y \in \mathfrak{G}$, *then* $xH_\rho y$ *in case* $y \in x\mathfrak{H}$.

THEOREM 1. *The relation H_ρ is a congruence relation for the loop $\mathfrak{G}$ if and only if*

$$\text{(i) } x \in (xh)\mathfrak{H}, \qquad \text{(ii) } (x\mathfrak{H})(y\mathfrak{H}) \subset (xy)\mathfrak{H}$$

for every $x, y \in \mathfrak{G}$ and $h \in \mathfrak{H}$.

PROOF. Let H_ρ be a congruence relation for $\mathfrak{G}$. For each $x \in \mathfrak{G}$ and $h \in \mathfrak{H}$ we have $xH_\rho(xh)$ by the definition of H_ρ. Since H_ρ is symmetric we have $(xh)H_\rho x$, $x \in (xh)\mathfrak{H}$. If also $y \in \mathfrak{G}$, $h_1 \in \mathfrak{H}$, then $yH_\rho(hy_1)$, and since H_ρ preserves multiplication we obtain $(xy)H_\rho(xh)(hy_1)$, $(xh)(yh_1) \in (xy)\mathfrak{H}$. Conversely, let $\mathfrak{H}$ satisfy (i) and (ii). If $x \in \mathfrak{G}$, choose $h \in \mathfrak{H}$ and we have $x \in (xh)\mathfrak{H} \subset x\mathfrak{H}$ by (i) and (ii). Thus H_ρ is reflexive. If also $y \in \mathfrak{G}$ and $xH_\rho y$, we have $y = xh_1$, and (i) yields $x \in (xh_1)\mathfrak{H} = y\mathfrak{H}$, $yH_\rho x$. Thus H_ρ is symmetric. If also $z \in \mathfrak{G}$ and $yH_\rho z$, we have $z = yh_2 = (xh_1)h_2 \in x\mathfrak{H}$ by (ii). Thus H_ρ is transitive and is an equivalence relation. The relations $xH_\rho y$, $zH_\rho w$ yield $(xz)H_\rho(yw)$ by (ii) and we conclude that H_ρ is a congruence relation for $\mathfrak{G}$.

Remark 1. If R is a congruence relation for a loop $\mathfrak{G}$, then the subset $\mathfrak{R} \equiv [x; xRe]$ is a *subloop* of $\mathfrak{G}$. For clearly $e \in \mathfrak{R}$ and $\mathfrak{R}$ is closed with

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[1] Numbers enclosed in brackets denote the references given at the end of the paper.