THE ROLE OF INTERNAL FAMILIES IN MEASURE THEORY

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1. **Introduction.** Theorem 4.7 below is an abstract formulation of a certain closed subset theorem[1] recently established by Randolph and myself. It has a wider range of application than similar abstractions due to Hahn[2] and to Saks.[3]

2. **Notation and terminology.** When H is a family of sets we agree that

$$\sigma(H) = \sum_{\beta \in H} \beta, \qquad \pi(H) = \prod_{\beta \in H} \beta.$$

A family R is said to be: *finitely additive* if $\sigma(H) \in R$ whenever H is a finite nonvacuous subfamily of R; *countably additive* if $\sigma(H) \in R$ whenever H is a countable nonvacuous subfamily of R; *finitely multiplicative* if $\pi(H) \in R$ whenever H is a finite nonvacuous subfamily of R; *countably multiplicative* if $\pi(F) \in R$ whenever F is a countable nonvacuous subfamily of R; *α complemental* if R is such a family of subsets of α that $\alpha - \beta \in R$ whenever $\beta \in R$.

If R is a family of sets we also agree that: R_σ is the family of all sets of the form $\sigma(H)$ where H is a countable nonvacuous subfamily of R; R_δ is the family of all sets of the form $\pi(H)$ where H is a countable nonvacuous subfamily of R; R_γ is the family of all sets of the form $\sigma(R) - \beta$ where $\beta \in R$; R^γ is the smallest $\sigma(R)$ complemental, countably additive family which contains R; R^δ is the smallest countably multiplicative, countably additive family which contains R.

DEFINITION 2.1. R is *internal* if and only if R_δ is finitely additive and $R_\gamma \subset R^\delta$.

REMARK 2.2 If R is the family of all closed subsets of a metric space then R is internal[4] and the members of R^γ are the Borel subsets of the space.

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[1] A. P. Morse and J. F. Randolph, *The ϕ rectifiable subsets of the plane*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 236–305, Theorem 3.7 together with the remarks which follow Theorem 3.4.

[2] H. Hahn, *Über die Multiplikation total-additiver Mengenfunktionen*, Annali della R. Scuola Normale Superiore Pisa (2) vol. 2 (1933) p. 437.

[3] S. Saks, *Theory of the integral*, Warsaw, 1937, p. 85.

[4] Since an open set is an R_σ.