# EQUIVALENCE OF A PROBLEM IN MEASURE THEORY TO A PROBLEM IN THE THEORY OF VECTOR LATTICES

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In connection with some other work of the author a question arises concerning the form of the general bounded linear[1] functional on certain vector lattices.[2] It is the purpose of this note to show that this question is completely equivalent to a question in measure theory which has been discussed and partially answered by Ulam [**2**].

Let $S$ be an abstract set and let $\mathfrak{F}$ be the vector lattice of all real-valued functions defined on $S$. For each $s_0$ in $S$ the function $F$ on $\mathfrak{F}$ such that $F(f)=f(s_0)$ for all $f$ in $\mathfrak{F}$ is clearly a linear functional. We shall call it the point functional belonging to $s_0$ or simply a point functional. Obviously every point functional and hence every finite linear combination of point functionals is bounded in the sense that it carries every bounded subset of $\mathfrak{F}$ into a bounded set of real numbers. Our question is as to whether every bounded linear functional on $\mathfrak{F}$ is a finite linear combination of point functionals. We shall show that this is the case if and only if there exists no countably additive measure $\alpha$ which is defined for all subsets of $S$, which is zero at points, which takes on only the values zero and one, and which does not vanish identically.

It is well known that every bounded linear functional on a vector lattice is a difference of non-negative linear functionals[3] and it is obvious that a non-negative linear functional is bounded. It follows that we need only consider non-negative linear functionals. Passages from a measure to a non-negative linear functional defined on a class of functions and vice versa are of frequent occurrence in mathematical literature. The proof of our theorem rests basically on the fact that when the methods used in effecting these passages are applied to the case at hand one obtains a natural one-to-one correspondence between the non-negative linear functionals on $\mathfrak{F}$ and the countably ad-

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[1] By a linear functional we mean a functional which is additive and homogeneous; that is, one which preserves linear combinations.

[2] See chap. 7 of [**1**] for definitions of the terms from the theory of vector lattices which we shall use. Numbers in brackets refer to the references cited at the end of the paper.

[3] This is proved on p. 115 of [**1**] for additive functionals and it is clear that an additive non-negative functional must be linear.