FOURIER SERIES WITH COEFFICIENTS IN A BANACH SPACE

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Let $f(t)$ be a function on (0, 1) to the complex Banach space B. Bochner has shown that the older theory of Fourier series carries over to functions of this character, but breaks down in the fundamental L^2 theory [1, pp. 273–276].[1]

Suppose $f(t)$ belongs to L^2 in the sense of Bochner [1]. Define

$$c_n = \int_0^1 f(t)e^{2\pi int}dt. \tag{1}$$

We should expect that the Parseval relation carries over, or at least that the Bessel inequality

$$\sum \|c_n\|^2 \leqq \int_0^1 \|f(t)\|^2 dt \tag{2}$$

is valid. This, however, is not the case; for suitable B we may have $\sum\|c_n\|^2 = \infty$ [1, pp. 275–276].

In this note we detect the root of the trouble by proving that for the validity of (2), B must possess a special character.

THEOREM. *If* (2) *is valid for all* $f(t)$ *in* L^2 *then* B *is unitary, and conversely.*

B is *unitary* if it admits a *scalar* product with the usual properties [3] (cf. the "normed ring" of Gelfand [2]).

The latter part of the theorem is trivial; we need only apply the classical proof with notational modifications [4, p. 58].

To establish the sufficiency suppose a and b are elements of B. Define

$$f(t) = \begin{cases} 2a, & (0, 1/2), \\ 2b, & (1/2, 1). \end{cases}$$

Then

$$\int_0^1 \|f(t)\|^2 dt = 2[\|a\|^2 + \|b\|^2]. \tag{3}$$

By (i) we have

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[1] Numbers in brackets refer to the references listed at the end of the paper.