PROOF. Apply Theorem J for the case $n=r$. We obtain for g_r an expression which differs from the one just written only in the fact that the terms $B_{r,r+1}f_{r+1}+B_{r,r+2}f_{r+2}+\cdots+B_{r,n}f_n$ are missing from its numerator. But the coefficients $B_{r,s}=0$ when $r<s$. Hence the two expressions are equal.

REMARK. The generalization of the method to orthonormalization with respect to a general norming or weight function p is obvious. One applies the process described to the functions $(p)^{1/2}f_i$ and obtains functions g_i ($\equiv$linear combinations of the $(p)^{1/2}f_i$) which are orthonormal with respect to the weight function unity. Dividing through by the common factor $(p)^{1/2}$ one forms functions $g_i(p)^{-1/2}$ which are orthonormal with respect to p.

CARNEGIE INSTITUTE OF TECHNOLOGY

VERTICES OF PLANE CURVES

S. B. JACKSON

1. **Introduction.** The Four-Vertex Theorem, proved first by Mukhopadhyaya [1],[1] states that on any oval, not a circle, there are at least four vertices, that is, extrema of the curvature. This result was extended by Fog [2] and Graustein [3] to any simple closed curve with continuous curvature. The discussion by Graustein makes it clear that the Four-Vertex Theorem is valid also for a very large number of non-simple curves. Indeed the class of curves having only two vertices is relatively quite small. The main object of the present paper is to characterize geometrically, as far as possible, the curves with just two vertices. It is thus a proof of the Four-Vertex Theorem by exclusion.

Since a curve with just two vertices consists of two arcs of monotone curvature, a study is made of such arcs (§2). The most useful fact is that this monotone character of an arc is invariant under direct circular transformations. The property that a point be a vertex of a curve is similarly invariant. This makes it possible to simplify many of the discussions by suitably chosen transformations. Monotone arcs are found to be essentially simple and possess a spiral character.

The existence of vertices on certain types of arcs is established (§4)

Presented to the Society, April 29, 1944; received by the editors February 15, 1944.

[1] Numbers in brackets refer to the Bibliography at the end of the paper.