ON CERTAIN ARITHMETICAL FUNCTIONS DUE TO G. HUMBERT

M. A. BASOCO

1. Introduction. G. Humbert has discussed, in a series of brief notes,[1] a certain class of entire functions with interesting arithmetical properties. These functions are defined, in an essentially unique manner, by certain functional equations. The Fourier series representations of the solutions of these equations are similar in form to those for the elliptic functions *snu*, *cnu*, *dnu*, and so on. They differ from these, however, in that their domain of validity extends throughout the entire complex plane (Z_∞ excluded) and moreover, in that their arithmetized forms involve *incomplete* numerical functions of the divisors of an integer.

In the present paper we extend somewhat the results of Humbert and obtain a relation between his functions and certain pseudo-periodic functions discussed elsewhere by the writer.[2] This relation is, in effect, embodied in a series of twelve identities; these are of some interest in that their arithmetical equivalents (paraphrases) are relatively simple and involve partitions related to the representations of an integer as the sum of five squares.

It is also pointed out that as an immediate consequence of the analytical form of Humbert's functions, it is possible to deduce a series of relations between the greatest integer function $E(x)$ and certain incomplete numerical functions of the divisors of an integer.

2. The functional equations. In what follows the notation is that ordinarily used in the theory of the elliptic theta functions.[3] The period $\pi\tau$ is such that $0<\arg\tau<\pi$.

The set of functional equations considered has the form

$$\text{(A)}\qquad \begin{aligned} h(z+\pi) &= (-1)^a h(z),\\ h(z+\pi\tau) &= (-1)^b h(z) + F_{ab}^{(\alpha)}(z), \end{aligned}$$

where a, b take the values zero or unity, and $F_{ab}^{(\alpha)}(z)$, to be defined presently, is an expression which involves the theta function $\vartheta_\alpha(z)$.

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[1] G. Humbert, *Sur quelques fonctions numeriques remarquables*, C. R. Acad. Sci. Paris vol. 158 (1914) pp. 220, 294, and 1841; vol. 163 (1916) p. 412.

[2] M. A. Basoco, Amer. J. Math. vol. 54 (1932) pp. 242–252.

[3] Whittaker and Watson, *Modern analysis*, Cambridge.