ALMOST ORTHOGONAL SERIES

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1. **Almost orthogonal series.** Let us consider an infinite sequence $\{\phi_n(x)\}$, $n=1, 2, \cdots$, of complex-valued functions of the real variable x, of class $L^2(a, b)$, normalized so that $\int_a^b|\phi_n(x)|^2dx=1$ for all n. Assume further that the sequence satisfies the following condition

$$\sum_{m,n}|a_{mn}|^2<\infty, \tag{1}$$

where $a_{mn}=\int_a^b\phi_m\bar{\phi}_n dx$ $(m\neq n;\ n,\ m=1, 2, \cdots)$, $a_{mn}=0$, $m=n$.

We wish to show that under the above conditions we have a Bessel inequality and an analogue of the Riesz-Fisher theorem.

THEOREM 1 (BESSEL'S INEQUALITY). *Under the above conditions, let $f(x)$ be a real-valued function belonging to $L^2(a, b)$, and $b_n=\int_a^b f\bar{\phi}_n dx$, then*

$$\sum_1^\infty|b_k|^2\leqq\int_a^b|f|^2dx\left[1+\left(\sum_{m,n}|a_{mn}|^2\right)^{1/2}\right].$$

We have

$$\sum_1^n|b_k|^2=\int_a^b f\left[\sum_1^n \bar{b}_k\bar{\phi}_k\right]dx.$$

Using Schwartz's inequality, this becomes

$$\begin{aligned}\sum_1^n|b_k|^2&\leqq\left[\int_a^b|f|^2dx\right]^{1/2}\left[\int_a^b\left[\sum_1^n b_k\phi_k\right]\left[\sum_1^n\bar{b}_k\bar{\phi}_k\right]dx\right]^{1/2}\\&\leqq\left[\int_a^b|f|^2dx\right]^{1/2}\left[\sum_1^n|b_k|^2+\sum_{1,1,k\neq l}^{n,n}b_k b_l a_{kl}\right]^{1/2}\\&\leqq\left[\int_a^b|f|^2dx\right]^{1/2}\left[\sum_1^n|b_k|^2\right.\\&\qquad\left.+\left\{\sum_{1,1}^{n,n}|b_k|^2|b_l|^2\right\}^{1/2}\left\{\sum_{1,1}^{n,n}|a_{kl}|^2\right\}^{1/2}\right]^{1/2}\\&\leqq\left[\int_a^b|f|^2dx\right]^{1/2}\left[\sum_1^n|b_k|^2\right]^{1/2}\\&\qquad\cdot\left[1+\left(\sum_{m,n}|a_{kl}|^2\right)^{1/2}\right]^{1/2}.\end{aligned}$$

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