SCHUR'S THEOREMS ON COMMUTATIVE MATRICES

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In 1905 I. Schur[1] proved that the maximum number $N(n)$ of linearly independent commutative matrices of n rows and columns is given by the formula $N(n)=[n^2/4]+1=\nu^2+1$ if $n=2\nu$ and $=\nu(\nu-1)+1$ if $n=2\nu-1$. Schur also determined the sets of linearly independent commutative matrices containing $N(n)$ elements. In this note we give a simpler derivation of Schur's results and an extension of these results from algebraically closed fields to arbitrary fields.

If $A_1, \cdots, A_{N(n)}$ is a set of linearly independent commutative matrices, the set $\mathfrak{A}$ of matrices $\sum A_i\phi_i$ where ϕ_i is arbitrary in the underlying field Φ is a commutative subalgebra containing the identity of the matrix algebra Φ_n. Hence $N(n)$ is the maximal dimensionality of commutative subalgebras of Φ_n. It is easy to see that $N(n)\geqq[n^2/4]+1$. For consider the set $\mathfrak{Z}_n$ of matrices

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \tag{1}$$

where if $n=2\nu$, A is arbitrary in Φ_ν and if $n=2\nu-1$, A is an arbitrary matrix of ν rows and $\nu-1$ columns. Thus $\dim \mathfrak{Z}_n=[n^2/4]$. It may be verified that $\mathfrak{Z}_n$ is a zero algebra. Hence the algebra $\mathfrak{B}_n$ obtained by adjoining 1 to $\mathfrak{Z}_n$ is a commutative algebra of dimensionality $[n^2/4]+1$. We remark also that if $n=2\nu-1$ we may replace $\mathfrak{Z}_n$ by the algebra $\overline{\mathfrak{Z}}_n$ of matrices of the form (1) in which A is an arbitrary matrix of $\nu-1$ rows and ν columns. We denote by $\overline{\mathfrak{B}}_n$ the extension of $\overline{\mathfrak{Z}}_n$ obtained by adjoining 1.

To prove that $N(n)\leqq[n^2/4]+1$ it suffices to assume that Φ is algebraically closed. For if $A_1, \cdots, A_{N(n)}$ are linearly independent and commutative in Φ_n, then they have these properties in Σ_n for any extension field Σ of the field Φ. Thus $N(n, \Phi)\leqq N(n, \Sigma)$. We shall therefore assume that Φ is algebraically closed. Let $\mathfrak{A}$ be a commutative subalgebra of Φ_n containing the identity and let N be the dimensionality of $\mathfrak{A}$ over Φ. We suppose first that $\mathfrak{A}$ is an indecomposable algebra of matrices. Then it is known that by replacing $\mathfrak{A}$ by a similar set we may suppose that the matrices of $\mathfrak{A}$ have the form

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[1] *Zur Theorie vertauschbaren Matrizen*, J. Reine Angew. Math. vol. 130 (1905) pp. 66–76.