A NOTE ON RIESZ SUMMABILITY OF THE TYPE e^{n^α}

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Recently I proved the following result in the case $r=2$ (Wang [4][1]).

Let $\sigma_n^{(r)}$ be the rth Cesàro mean of the series $\sum_{n=0}^\infty a_n$. If $\sigma_n^{(r)}-s=o(n^{-r\alpha})$ $0<\alpha<1$, as $n\rightarrow\infty$, where r is a positive integer, and $a_n>-Kn^{\alpha-1}$, the series converges to sum s.

For the case $r=1$ this result has been established by Boas [1]. His argument, however, does not seem to be applicable in any simple way to the general case.

The object of this note is to prove a theorem on Riesz summability of type e^{n^α}, and then to deduce the result above from a Tauberian theorem of Hardy [2].

Let us put $C_\tau(\omega)=a_0e^{\tau\omega^\alpha}+\sum_{n<\omega}(e^{\omega^\alpha}-e^{n^\alpha})^\tau a_n$. A series $\sum_{n=0}^\infty a_n$ is said to be summable (e^{n^α}, τ) to the sum s if

$$C_\tau(\omega) = se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}). \tag{1}$$

The result by Hardy which is to be called upon is the following: If the series $\sum_{n=0}^\infty a_n$, with terms $a_n\geqq -Kn^{\alpha-1}$, $0<\alpha<1$, is summable (e^{n^α}, τ), it is convergent. We shall now prove the following theorem.

THEOREM. *If $\sigma_n^{(r)}-s=o(n^{-r\alpha})$, $0<\alpha<1$, as $n\rightarrow\infty$, the series $\sum_{n=0}^\infty a_n$ is summable (e^{n^α}, τ) to the sum s, where $\tau>r/(1-\alpha)$.*

To prove this let $\beta_n=(e^{\omega^\alpha}-e^{n^\alpha})^\tau$, $\Delta\beta_n=\beta_n-\beta_{n+1}$, $\Delta^{r+1}\beta_n=\Delta^r\beta_n-\Delta^r\beta_{n+1}$ and

$$s_n^{(r)}=\sum_{\nu=0}^n\binom{n-\nu+r}{n-\nu}a_\nu,$$

$m=[\omega]$. Then, by successive Abel's transformations we have

$$\begin{aligned} C_\tau(\omega) &= a_0e^{\tau\omega^\alpha} + \sum_{n=1}^m \beta_n a_n \\ &= a_0e^{\tau\omega^\alpha} + \sum_{n=1}^{m-r+1} s_n^{(r)}\Delta^{r+1}\beta_n + \sum_{i=0}^r s_{m-i}^{(i)}\Delta^i\beta_{m-i} - \sum_{i=0}^r s_0^{(i)}\Delta^i\beta_1 \qquad (2)\\ &= a_0e^{\tau\omega^\alpha} + J_1 + J_2 - J_3. \end{aligned}$$

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[1] Numbers in brackets refer to the references listed at the end of the paper.