ON STRONG SUMMABILITY OF A FOURIER SERIES

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Let $s_n(x) = a/2\sum_{\nu=1}(a_\nu \cos \nu x + b_\nu \sin \nu x)$ be the partial sum of the Fourier series of an integrable periodic function $f(t)$ of period 2π, and let $\phi(t) = \{f(x+t)+f(x-t)-2s\}/2$. We shall establish the following result (Hardy-Littlewood [1][1]).

THEOREM. *If*

$$\int_0^t |\phi(u)| \{1 + \log^+ |\phi(u)| \} du = o(t), \qquad \text{as } t \to 0, \tag{1}$$

then $\sum_{\nu=0}^n |s_\nu(x) - s|^2 = o(n \log \log n)$, *as* $n \to \infty$.

To prove this theorem we require the following lemmas.

LEMMA 1. *If*

$$\int_0^t |\phi(u)| du = o(t), \qquad \text{as } t \to 0, \tag{2}$$

then

$$\sum_{\nu=0}^n | s_\nu(x) - s|^2 = \frac{1}{\pi^2}\int_{1/n}^{\delta} \frac{\phi(t)}{t^2} dt \int_{1/n}^{t} \phi(u) \frac{\sin n(u-t)}{u-t} du + o(n).$$

PROOF. By (2), for $\nu \leqq n$,

$$s_\nu(x) - s = \frac{2}{\pi}\int_{1/n}^{\delta} \phi(t) \frac{\sin \nu t}{t} dt + o(1).$$

Hence

$$\begin{aligned}
\sum_{\nu=0}^n | s_\nu(x) - s|^2 &= \frac{4}{\pi^2}\int_{1/n}^{\delta}\int_{1/n}^{\delta} \frac{\phi(u)\phi(t)}{ut}\left\{ \sum_{\nu=1}^n \sin \nu t \sin \nu u\right\} dudt + o(n) \\
&= \frac{2}{\pi^2}\int_{1/n}^{\delta}\int_{1/n}^{\delta} \frac{\phi(u)\phi(t)}{ut} \frac{\sin n(u-t)}{u-t} dudt \\
&\quad + \frac{2}{\pi^2}\int_{1/n}^{\delta}\int_{1/n}^{\delta} \frac{\phi(u)\phi(t)}{ut} \frac{\sin n(u+t)}{u+t} dudt + o(n) \\
&= J_1 + J_2 + o(n).
\end{aligned} \tag{3}$$

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[1] Numbers in brackets refer to the references listed at the end of the paper.