A THEOREM ON THE UNIT GROUPS OF SIMPLE ALGEBRAS

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1. Introduction. Let k be an algebraic number field of finite degree m and let A be a normal simple algebra of degree n, order n^2, over k. Our object is to prove the following theorem.

THEOREM. *If A is an R-algebra, that is, if $n>2$ or at least one infinite prime place of k is unramified in A when $n=2$, then any two distinct maximal orders of A have distinct groups of units.*

There are essential arithmetical differences between algebras which satisfy the R-condition (R-algebras) and those which do not, especially with regard to class-number properties (Eichler [1, 2, 3]).[1] The meaning of the R-condition in the case $n=2$ is as follows. Both k and A are simple algebras over the field k_0 of rational numbers, of orders m and $4m$, respectively, over k_0. Suppose k_0 is extended to the field k_1 of real numbers. Then the extended algebra $k\times k_1$ is the direct sum of fields, each of which is isomorphic either to k_1 or to the field k_2 of complex numbers. This decomposition of $k\times k_1$ involves the decomposition of $A\times k_1$ into a direct sum of simple algebras over k_1, the centers of which are the corresponding summands of $k\times k_1$. Each summand of $A\times k_1$ is either (1) a matrix algebra of degree 2 over k_1, (2) a matrix algebra of degree 2 over k_2, or (3) the division algebra of quaternions over k_1. With each summand of $k\times k_1$ is associated an infinite prime place of k which is said to be ramified or unramified in A according as the corresponding summand of $A\times k_1$ is (3) or is either (1) or (2). The R-condition for $n=2$ is thus equivalent to requiring that not all summands of $A\times k_1$ be (3), in other words, that A over k is not a totally definite quaternion algebra. The condition is in general indispensable in our theorem. For example, the unit groups of all maximal orders in certain definite quaternion algebras over k_0 consist of the units ± 1 only.

The proof of the theorem will be based on the following Hilfssatz due to Eichler [3, p. 239, Hilfssatz 9].

Let A be an R-algebra, let $\mathfrak{O}$ be a maximal order of A, and let $\mathfrak{F}$ be a two-sided $\mathfrak{O}$-ideal. If $\mathcal{A}$ is an element of $\mathfrak{O}$ whose reduced norm $N(\mathcal{A})$ is congruent modulo $\mathfrak{F}$ to a unit of k, then $\mathfrak{O}$ contains a unit $\mathcal{E}\equiv\mathcal{A}$ modulo $\mathfrak{F}$.

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[1] Numbers in brackets refer to the references cited at the end of the paper.