CLUSTER POINTS OF SUBSEQUENCES[1]

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In the preceding paper [1][2] Buck defines a class of "subsequences" of a multiple sequence and shows that "almost all" of such subsequences have certain properties. This note is essentially based on a different choice of the definition of "subsequences"; that is, this paper and [1] are generalizations in different directions of a preceding paper by Buck and Pollard (reference 2 of [1]). In this discussion countability is the important property of the index systems such as the integers underlying the simple sequences or the n-tuples of integers underlying the multiple sequences. Countability is a slightly stronger condition than is necessary since the results will be shown to hold as well for functions of n variables as for multiple sequences; some other special cases are mentioned at the end of this paper. Also I modify Buck's approach by considering cluster points in neighborhood spaces rather than limit points in convergence spaces [3]. It may be mentioned that even for multiple sequences Theorems 1 and 2 of these papers are independent since Buck's set of "subsequences" is a set of measure zero in the set of "subsequences" considered here; my Theorem 3 contains the corresponding theorem of [1] as a special case. Lemma 1 and its corollary, Lemma 3, are the fundamental results on which the theorems rest; Lemma 3 is the generalization appropriate to this paper of the lemma in §3 of [1].

1. **Preliminaries.** If R is any set, a product measure can be defined in the set of characteristic functions of subsets of R [1, footnote 2] and this in turn induces a measure $|\cdots|$ for subsets of the set $\mathcal{E}$ of all subsets E of R; this measure is non-negative, completely additive, and (if R is infinite) takes all values between 0 and 1 inclusive; its other principal characteristic is that if $r_1, \cdots, r_k \in R$, then[3] $\{E| \text{ no } r_i \in E\}$ is of measure 2^{-k}; hence if E_0 is an infinite subset of R and $A = \{E | E \cap E_0 \text{ is empty}\}$, $|A| = 0$.

An *index system* $\mathfrak{R} = (R, \geqq)$ is a set R and a binary relation $\geqq$ such that $\geqq$ is transitive and every element r_0 has a successor $r_1 > r_0$ such that $r_0 \not> r_1$. (In the language of [4] $\mathfrak{R}$ is oriented and has no terminal

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[1] Considerations suggested by the preceding paper of R. C. Buck.

[2] Numbers in brackets refer to the Bibliography at the end of the paper.

[3] The usual notation of $\cup$ and $\cap$ will be used for union and intersection of sets; $\{p|P\}$ will mean the set of all p having the property P.