ON THE EQUATION $\chi\alpha=\gamma\chi+\beta$ OVER AN ALGEBRAIC DIVISION RING

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1. Introduction and notation. The main purpose of this paper is to give necessary and sufficient conditions in order that the equation

$$\chi\alpha = \gamma\chi + \beta \tag{1}$$

have a solution χ over an algebraic division ring. In case a solution exists, it is given explicitly if it is unique; otherwise, a method of obtaining one of the solutions is given. The application of the results to a quaternion algebra is discussed in the final section.

Let R be a division ring algebraic over its separable[1] center F, and λ a commutative indeterminate over R. Using the notation of Ore,[2] a polynomial $a(\lambda) \in R[\lambda]$ of degree n,

$$a(\lambda) = \alpha_n\lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0, \tag{2}$$

will be called reduced if $\alpha_n = 1$. The unique reduced polynomial $m(\lambda) \in F[\lambda]$ of minimum degree for which $m(\alpha) = 0$ will be labelled $m_\alpha(\lambda)$. It is apparent that $m_\alpha(\lambda)$ is irreducible over $F[\lambda]$. The ring of all elements of R which commute with α will be denoted by R_α.

The substitution of an element of R for λ in the polynomial (1) is not well defined, as λ commutes with elements of R, whereas the elements of R do not all commute among themselves. However, unilateral substitution is well defined. We shall use the symbol $a^r(\beta)$ to mean that β has been substituted for λ on the right in (2), so that

$$a^r(\beta) = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_0. \tag{3}$$

Left substitution is defined similarly—as there is a complete duality between left and right substitution in our case, we shall discuss right substitution only. If $a^r(\beta) = 0$, β is called a right root of $a(\lambda)$. The notation $a(\lambda) \mid^r b(\lambda)$ is used to mean that $a(\lambda)$ is a right factor of $b(\lambda)$. As is well known, β is a right root of $a(\lambda)$ if and only if $(\lambda-\beta) \mid^r a(\lambda)$.

2. Preliminary lemmas. A division algorithm exists over $R[\lambda]$. The particular case of interest here is given by

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[1] That is, no irreducible polynomial in $F[\lambda]$ has a multiple root in R.

[2] O. Ore, *Theory of noncommutative polynomials*, Ann. of Math. vol. 34 (1933) pp. 481–508.