ON MINIMUM CIRCUMSCRIBED POLYGONS

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This paper contains the proofs of two theorems on the n-gon M_n of minimum area circumscribed about a convex region R in the plane. Theorem 1 shows that the area of M_n is a convex function of n and Theorem 3 shows that if R is symmetric about a point there exists an M_{2n} which is also symmetric. The corresponding theorems on inscribed polygons are also given. These theorems were conjectured by R. B. Kershner.

The symbols a and b with subscripts will be used to represent the sides of circumscribed polygons or the vertices of inscribed polygons. It will be convenient to replace the circular order of the sides (vertices) $a_0, a_1, \cdots, a_{n-1}$ of a polygon by an artificial linear ordering $a_0 < a_1 < \cdots < a_{n-1} < a_n$, where a_n represents the same side (vertex) as a_0. The order of the sides (vertices) of circumscribed (inscribed) polygons is established by the order of the contact points (vertices) on the boundary of the convex region.

LEMMA 1. *Let $A_n = a_0 a_1 \cdots a_{n-1}$ and $B_m = b_0 b_1 \cdots b_{m-1}$ be two polygons circumscribed about the convex region R and let either* (1) $a_0 \leqq b_0 < b_1 < a_1 \leqq a_{r-1} \leqq b_{s-1} < b_s < a_r$ *or* (2) $a_0 \leqq b_0 < b_1 < a_1 \leqq b_{s-1} \leqq a_{r-1} < a_r < b_s$. *Let $C_{n-r+s} = a_0 b_1 b_2 \cdots b_{s-1} a_r \cdots a_{n-1}$ and $D_{m-s+r} = b_0 a_1 a_2 \cdots a_{r-1} b_s \cdots b_{m-1}$. Then the areas satisfy the inequality $A + B \geqq C + D$ and there is equality if and only if $a_0 = b_0$ and $a_{r-1} = b_{s-1}$.*

PROOF. The common part of A and B is the common part of C and D. The remaining part of $A+B$ is the remaining part of $C+D$ together with the areas of the quadrilaterals $a_0 a_1 b_0 b_1$ and $a_{r-1} a_r b_{s-1} b_s$. Equality holds if and only if both these areas are zero, that is, if $a_0 = b_0$ and $a_{r-1} = b_{s-1}$.

LEMMA 2. *If M_n is an n-gon of maximum area inscribed in the convex region R, the vertices of M_n are contact points of the sides of a circumscribed polygon.*

PROOF. If $M_n = a_0 a_1 \cdots a_{n-1}$, the line through a_1 parallel to the line $a_0 a_2$ is a supporting line of R, for otherwise M_n would not have maximum area. Supporting lines determined similarly at all vertices of M_n are seen to form a circumscribed polygon.

LEMMA 3. *Let $A_n = a_0 a_1 \cdots a_n$ and $B_m = b_0 b_1 \cdots b_m$ be two polygons*

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