A NOTE ON SUBSEQUENCES

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It is well known that a sequence is convergent if all of its subsequences are convergent. The purpose of this note is to show that this may be extended to a result of great generality.

THEOREM. *A sequence is convergent if there exists a regular matrix summability method T, which sums every subsequence of the given sequence.*

Thus, for example, if every subsequence of a given sequence is Cesàro summable of any order, then the sequence is convergent. The general theorem depends upon the following well known result of H. Steinhaus.[1]

LEMMA. *For any regular summability method T, given by the matrix (A_{nm}), there exists a sequence of* 0*'s and* 1*'s not summable T.*

To establish the theorem, we suppose that the given sequence $\{S_n\}$ is divergent; we then show that, for any T, we can choose a subsequence $\{S_{r_n}\}$ not summable T. We may obviously restrict ourselves to bounded sequences. Set $S_n' = (S_n - \liminf S_n)/(\limsup S_n - \liminf S_n)$. We observe that $\liminf S_n' = 0$, $\limsup S_n' = 1$, so that we may choose two disjoint subsequences

$$\{S'_{\alpha_n}\} \quad \text{and} \quad \{S'_{\beta_n}\}$$

which converge to 1 and 0 respectively.

We next construct a "replica" $\{S_n^*\}$ of the sequence $\{S_n'\}$ by defining

$$S_n^* = \begin{cases} 1 & \text{if} \quad n = \alpha_k \quad \text{for some } k. \\ 0 & \text{if} \quad n = \beta_k \quad \text{for some } k. \\ S_n' & \text{otherwise.} \end{cases}$$

To justify the use of the term replica, we observe that $\{S_n^*\}$ and $\{S_n'\}$ have the same limit points, that $\lim (S_n' - S_n^*) = 0$, and that in general the two sequences behave alike.

Among the terms of $\{S_n^*\}$, there are an infinite number of 1's and of 0's; by the Steinhaus Lemma, for any T, we can choose a subse-

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[1] Hugo Steinhaus, *Kitka słów o uogólnieniu pojecia granicy* (Some remarks on the generalization of limit), Prace Matematyczno-Fizyczne vol. 22 (1911) pp. 121–134 (p. 129).