ON CERTAIN PAIRS OF SURFACES IN ORDINARY SPACE

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1. **Introduction.** In a recent paper[1] Jesse Douglas has proposed and solved the following problem: To determine the form of the linear element of a surface in ordinary space upon which exists a family of ∞^2 curves possessing two properties: (1) The angular excess of any triangle ABC formed by curves of the family $\mathcal{J}$ is proportional to the area of the triangle:

$$\mathcal{E} = A + B + C - \pi = k\mathcal{A}, \tag{1}$$

where k denotes a constant; (2) The curves of $\mathcal{J}$ are a *linear* system; that is, a point transformation exists which converts them into the straight lines of a plane. It is natural to inquire what class of surfaces we shall obtain if, instead of using property (2), we make the less specific demand that a point transformation exists which converts the curves of $\mathcal{J}$ into the *geodesics* of another surface. Here we have found certain pairs of surfaces S and S_1 which furnish the complete solution of our generalized problem. According to whether the constant k is zero or not, the linear elements of S and S_1 take different types, whose derivation constitutes the purpose of the present paper.

2. **Conditions for the property $\mathcal{E} = k\mathcal{A}$.** As was shown by Douglas,[2] the necessary and sufficient conditions that every curve of a family $\mathcal{J}$ upon a surface S should have the property $\mathcal{E} = k\mathcal{A}$ can be expressed by the relation

$$ds/\rho = Pdu + Qdv, \tag{2}$$

where $1/\rho$ is the geodesic curvature of the curve and P, Q obey the condition

$$Q_u - P_v = (k - K)W. \tag{3}$$

For the subsequent discussion it is convenient to consider both surfaces S and S_1, wherein the curves of $\mathcal{J}$ upon S correspond to the geodesics of S_1. Let (u, v) be general coordinates of the corresponding points on these surfaces, so that the first fundamental form of S is

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[1] J. Douglas, *A new special form of the linear element of a surface*, Trans. Amer. Math. Soc. vol. 48 (1940) pp. 101–116.

[2] Douglas, loc. cit., p. 108.