A NOTE ON DIFFERENTIAL POLYNOMIALS

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The following theorem indicates to what extent the expression of a differential polynomial[1] G as an element of the differential ideal determined by F is unique.

THEOREM I. *Let $F\neq 0$, C_0, $C_1, \cdots, C_s$ be differential polynomials in the unknowns $y_1, \cdots, y_n$ with coefficients in an abstract differential field $\mathcal{J}$. Let $F^{(i)}$ be the ith derivative of F and let*

$$C_0F + C_1F' + \cdots + C_sF^{(s)} \tag{1}$$

be identically zero. Then each C_i is in the perfect ideal generated by F.[2]

We need merely show that any solution $y_j=\bar{y}_j$ $(j=1, \cdots, n)$, in any extension $\mathcal{J}_1$ of $\mathcal{J}$, of the form F is a solution of each C_i.[3] Since this is true if F has no solutions, we may assume that F effectively involves the unknowns. Make the substitution $y_j=z_j+\bar{y}_j$ in (1). Let A consist of the terms of F of lowest degree in the z_j and their derivatives. Collecting terms of the same degree, we see that

$$C_0(\bar{y})A + \cdots + C_s(\bar{y})A^{(s)} = 0, \tag{2}$$

where $C_i(\bar{y})$ is the element of $\mathcal{J}_1$ obtained by substituting $y_j=\bar{y}_j$ $(j=1, \cdots, n)$ in C_i. Let A be of order $p\geqq 0$ in some z_k which it effectively involves, let $z_{k,m}$ be the mth derivative of z_k, and let S be the partial derivative of A with respect to $z_{k,p}$. For $i>0$, $A^{(i)}$ can be written as $Sz_{k,p+i}+B_i$, where B_i is some form of order less than $p+i$ in z_k. Now (2) becomes

$$C_s(\bar{y})Sz_{k,p+s} + D = 0 \tag{3}$$

where D has order less than $p+s$ in z_k. Hence $C_s(\bar{y})=0$. In turn $C_{s-1}, \cdots, C_0$ must vanish for $y_j=\bar{y}_j$ as desired.

Using the ideas of the above proof together with a uniqueness result of J. F. Ritt,[4] one can very easily prove the following generalization.

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[1] For definitions of differential fields, polynomials, and ideals, see H. W. Raudenbush, Ann. of Math. (2) vol. 34 (1933) pp. 509–517.

[2] For a result analogous to Theorem I for ordinary polynomials, see Satz 1 of E. Lasker, *Zur Theorie der Moduln und Ideale*, Math. Ann. vol. 60 (1905) pp. 20–116.

[3] H. W. Raudenbush, Trans. Amer. Math. Soc. vol. 36 (1934) pp. 361–368.

[4] *On singular solutions* $\cdots$, Ann. of Math. vol. 37 (1936) pp. 552–617, §§1–3.