ON SEQUENCES OF POLYNOMIALS AND THE DISTRIBUTION OF THEIR ZEROS

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The first results on this subject are due to Laguerre (1882); they were generalized to a remarkable degree by Pólya and in a joint paper by Lindwart and Pólya. I quote the following theorems [2].[1]

THEOREM 1. *If a sequence of polynomials*

$$(1) \qquad P_n(z) = 1 + \sum_1^n c_{n\nu} z^\nu = \prod_\nu (1 - z z_{n\nu}^{-1})$$

converges uniformly in a circle $|z| < R$, *and if for some integer* k

$$(2) \qquad \sum_1^n |z_{n\nu}|^{-k} < M, \qquad M \text{ independent of } n,$$

then the sequence (1) *converges uniformly in every finite domain to an entire function* $F(z)$ *which is the product of a function of genus at most* $k-1$ *and of* $e^{\gamma z^k}$, γ *a constant.*

THEOREM 2. *If the sequence* (1) *converges uniformly in a circle* $|z| < R$, *and if the roots* $z_{n\nu}$ *lie in the half-plane* $\Re z \geqq 0$ *for each* n, *then the sequence* (1) *converges uniformly in every finite domain to an entire function* $F(z)$ *which is at most of genus* 2, *and the roots* z_ν *of* $F(z)$ *satisfy* $\sum |z_\nu|^{-2} < \infty$.

While in Theorem 1 the assumption of uniform convergence could be replaced by convergence at infinitely many points with a finite limit point and by boundedness of the sequences: $|c_{n1}|, \cdots, |c_{n\,k-1}|$, $n=1, 2, \cdots$, the deduction of Theorem 2 required uniform convergence in $|z| < R$. We give here a new proof for Theorem 2 with a weaker hypothesis assuming instead of uniform convergence only convergence at infinitely many points in some finite domain and boundedness of the sequences $|c_{n1}|$, $|c_{n2}|$. We further generalize the assumption on the location of the zeros (following a similar remark of Weisner [5]), assuming only that the zeros of $P_n(z)$ lie in a half-plane containing the origin on its boundary, but otherwise varying with n. Finally we extend the results to certain sequences of entire functions.

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[1] Numbers in brackets refer to the bibliography at the end of this paper.