SOME IMBEDDING THEOREMS AND CHARACTERIZATION PROBLEMS OF DISTANCE GEOMETRY

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Introduction. The systematic development of abstract distance geometry was initiated by Menger's *Untersuchungen* [6][1] of 1928. The field opened up by these pioneering papers has been extensively cultivated during the past fourteen years with the result that not only have the boundaries of the subject been extended far beyond what was envisaged a decade and a half ago, but the territory gained has been regained and consolidated by new methods that attain their objectives more easily. It is the purpose of this paper to present some of these consolidations (Part I) and extensions (Part II)—at hand or in progress—and to point out a few promising regions for future exploration.

PART I

1. **Preliminary definitions.** A *distance space*, in its most general aspects, arises upon associating with each ordered pair p, q of elements of a "point" set an element pq of a "distance" set, the association being conditioned only by certain very simple rules. If, in particular, the distance set is the set of non-negative real numbers, $pq = qp$, while $pq = 0$ if and only if $p = q$, one obtains the class $\{\Sigma\}$ of *semimetric spaces*. A semimetric space is *metric* provided that for each three of its points p, q, r the triangle inequality $pq + qr \geqq pr$ is satisfied.

We shall have frequent occasion to consider the determinant

$$D(p_1, p_2, \cdots, p_k) = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & p_1p_2^2 & \cdots & p_1p_k^2 \\ 1 & p_2p_1^2 & 0 & \cdots & p_2p_k^2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & p_kp_1^2 & p_kp_2^2 & \cdots & 0 \end{vmatrix},$$

formed for k points $p_1, p_2, \cdots, p_k$ of a semimetric space. With its use the triangle inequality takes the symmetric form $D(p, q, r) \leqq 0$.

Two distance spaces (or subsets of the same distance space) are

An address delivered before the Chicago meeting of the Society on April 17, 1942, by invitation of the Program Committee; received by the editors May 25, 1942.

[1] See references at end of paper.