ON THE AVERAGE NUMBER OF REAL ROOTS OF A RANDOM ALGEBRAIC EQUATION

M. KAC

1. **Introduction.** Consider the algebraic equation

$$X_0 + X_1x + X_2x^2 + \cdots + X_{n-1}x^{n-1} = 0, \tag{1}$$

where the X's are independent random variables assuming real values only, and denote by $N_n = N(X_0, \cdots, X_{n-1})$ the number of real roots of (1). We want to determine the mean value (mathematical expectation = m.e.) of N_n when all X's have the same normal distribution with density

$$e^{-u^2}/\pi^{1/2}. \tag{2}$$

This problem was treated by Littlewood and Offord[1] who also considered the cases when the X's are uniformly distributed in $(-1, 1)$ or assume only the values $+1$ and -1 with equal probabilities. Littlewood and Offord obtain in each case the estimate

$$\text{m.e.} \{N_n\} \leqq 25(\lg n)^2 + 12 \lg n, \qquad n \geqq 2000.$$

In our case of normally distributed X's we shall be able to prove the exact formula

$$\text{m.e.} \{N_n\} = \frac{4}{\pi}\int_0^1 \frac{[1 - n^2[x^2(1 - x^2)/(1 - x^{2n})]^2]^{1/2}}{1 - x^2} dx \tag{3}$$

and then obtain the asymptotic relation

$$\text{m.e.} \{N_n\} \sim (2/\pi) \lg n \tag{4}$$

and the estimate

$$\text{m.e.} \{N_n\} \leqq (2/\pi) \lg n + 14/\pi, \qquad n \geqq 2. \tag{5}$$

In case the X's are not normally distributed (but all have the same distribution with standard deviation 1) one can still prove (4). The necessary limiting processes can then be carried out by using the central limit theorem of the calculus of probability and, as one may expect, the computations will be quite lengthy. On the other hand they will contribute relatively little to the general picture and, what

Presented to the Society, September 10, 1942; received by the editors July 2, 1942.

[1] J. London Math. Soc. vol. 13 (1938) pp. 288–295. No proofs are given in this paper, and the present author was unable to find them anywhere in print.