THEOREM 11. *If* $(a, b) = 1$, *and* $m$ *is the product of the distinct prime factors of* $n$, *then*

$$(n^3/m) \mid \phi(a^{2n} - b^{2n}).$$

CORNELL UNIVERSITY

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# APPLICATIONS OF TRANSITIVITIES OF BETWEENNESS IN LATTICE THEORY[1]

M. F. SMILEY AND W. R. TRANSUE

**Introduction.** This paper solves three characterization problems for lattices[2] [1]. Problem I is to characterize those metric spaces [2] into which lattice operations which are consistent with the given metric [1, p. 41] may be introduced. Problem II is to characterize those members of a rather general class of abstract systems which are modular lattices, while Problem III consists in the characterization of lattices in an even larger class of abstract systems.

Problem I has already been solved by V. Glivenko [3]. He showed that the property: "Among those elements metrically between [4, p. 76; 2] two elements $a$ and $b$, the element $a \cup b$ is farthest from $O$," and its dual characterize those metric spaces which are also metric lattices with the same metric and least element $O$. Our approach to Problem I is through the existence of certain metric singularities [2, p. 47] in every metric lattice. Our solution also involves certain five point transitivities [5, Part I] of metric betweenness. The abstract system involved in Problem II (Problem III) is a wide generalization of the concept of a metric space—so general, in fact, that it also includes the concept of a modular lattice (lattice). We find it not difficult to extend the ideas essential to our solution of Problem I to give analogous solutions of Problems[3] II and III. Briefly, our results consist in characterizing the three important systems: metric

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[1] This paper is an expansion of our note *Metric lattices as singular metric spaces* which was presented to the Society on December 29, 1941, and which was to appear in Bull. Amer. Math. Soc.

[2] Numbers in square brackets refer to the list of references at the end of the paper.

[3] It would be interesting to characterize metric spaces among our general systems. This problem appears difficult to us, and we make no attempt to solve it.