THE ELEMENTARY DIVISOR THEOREM FOR CERTAIN RINGS WITHOUT CHAIN CONDITION

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1. **Introduction.** The Elementary Divisor Theorem is known to hold in principal ideal rings and in rings with Euclidean algorithm.[1] It is an open question whether it holds in every *Prüfer ring*, that is, in every domain of integrity in which every ideal with finite basis is a principal ideal.[2] (A Prüfer ring can also be characterized as a domain of integrity in which the greatest common factor of any finite number of elements can be represented as a linear combination of these elements.) The Elementary Divisor Theorem will here be proved for what will be called "adequate" rings. They are special Prüfer rings, but not restricted by any equivalent to a chain condition, so that they comprise considerably more than just the principal ideal rings.[3]

2. **Definition of adequate rings.** Let R be a domain of integrity, a, b in R, and $a \neq 0$. By a *relatively prime part* of a with respect to b, written $RP(a, b)$, we shall understand a factor a_1 of a such that, if $a = a_1 \cdot a_2$,

(i) $(a_1, b) = 1$,

(ii) $(a_3, b) \neq 1$ for any non-unit factor a_3 of a_2.[4]

$RP(a, b)$ may or may not exist; if it does, it is, in a sense, a largest factor of a that is relatively prime to b.

We now define R to be an *adequate ring* if

(i) R is a Prüfer ring,

(ii) $RP(a, b)$ exists for all a, b in R with $a \neq 0$.

3. **Relationship to Prüfer rings and principal ideal rings.** By definition every adequate ring is a Prüfer ring. On the other hand, every

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[1] B. L. van der Waerden, *Moderne Algebra*, vol. 2, Berlin, 1931, p. 122. For further reference, see the papers by J. H. M. Wedderburn (J. Reine Angew. Math. vol. 167 (1932)), N. Jacobson (Ann. of Math. (2) vol. 38 (1937)), and O. Teichmüller (Preuss. Akad. Wiss. Sitzungsber. 1937).

[2] Rings of this kind were considered by H. Prüfer in *Untersuchungen über Teilbarkeitseigenschaften in Körpern*, J. Reine Angew. Math. vol. 168 (1932).

[3] My thanks are due to Professor Reinhold Baer for several helpful suggestions.

[4] Notation throughout this paper: "$(a, b, \cdots)$" for "greatest common factor of $a, b, \cdots$," "$a \mid b$" for "a is a factor of b," "$a = b$" for "a equals b except possibly for a unit factor." (This last convention serves the purpose of replacing statements about principal ideals simply by statements about their generating elements.)