RECURRENCE FORMULAS FOR CERTAIN DIVISOR FUNCTIONS

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Numerical functions giving the excess of the number of divisors of n of one sort over the number of divisors of a second sort were introduced over a century ago from the theory of binary quadratic forms and from the theory of elliptic functions. Later a systematic discussion of several of these functions was made by J. W. L. Glaisher, who, by means of theta function identities, found recurrences with gaps for these functions. The purpose of this note is to point out that these functions are sums over divisors of certain periodic Lucas functions and that sums of Lucas functions in general satisfy triangular number recurrence relations as given by (6) below. This formula is specialized in several ways, first to periodic Lucas functions, then to degenerate functions, and then to the more typical cases related to the Pell equation and Fibonacci's series.

Formula (6) may be obtained in the following elementary manner from the famous triple product identity of Gauss and Jacobi.[1]

$$(1)\qquad \prod_{\nu=1}^{\infty}\left\{(1-x^{2\nu})(1+ax^{2\nu-1})(1+a^{-1}x^{2\nu-1})\right\} = \sum_{n=-\infty}^{\infty} a^n x^n .$$

If in (1) we set $x=t^{1/2}$ and $a=-\alpha t^{1/2}$ we obtain at once

$$(2)\quad (1-\alpha)\prod_{\nu=1}^{\infty}\left\{(1-t^{\nu})(1-\alpha t^{\nu})(1-\alpha^{-1}t^{\nu})\right\} = \sum_{n=-\infty}^{\infty}(-\alpha)^{n+1}t^{n(n+1)/2}.$$

If now we introduce θ by $\alpha=e^{2\theta i}$ and write

$$(3)\qquad P(\theta,t) = \prod_{\nu=1}^{\infty}\left\{(1-t^{\nu})(1-2t^{\nu}\cos 2\theta+t^{2\nu})\right\},$$

then, on combining the terms on the right of (2) for $n=r-1$ and $-r$,

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[1] K. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum*, Königsberg, 1829, §64; *Gesammelte Werke*, Berlin, 1881, vol. 1 pp. 232–234. Other elementary proofs of (1) (not depending on elliptic functions) have been given by A. Cauchy, C. R. Acad. Sci. Paris vol. 17 (1843) pp. 523–531; *Oeuvres*, vol. 8, pp. 42–50; A. Enneper, *Elliptische Functionen*, Halle, 1876, pp. 74–77; J. Tannery and J. Molk, *Éléments de la theorie des fonctions elliptiques*, Paris, 1896, vol. 2 pp. 10–13; G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford, 1938, pp. 280–281.